Bootstrapping Math Stuff

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1 Introduction

This was originally a two page note about basic ideas you should be comfortable with to make math a little easier. The particular math I had in mind was Walter Rudin's *Principles of Mathematical Analysis*. But the note just kept on getting bigger as I was typing it up.

People sometimes think of math as being a science, but really it's also a lot like engineering and computer programming in the sense that you build a big structure from a relatively small collection of basic ideas. Math is also an *art*. Good and interesting math involves creating a journey. The journey begins in some location or context within this structure, and then builds or points out something interesting. Hidden connections are revealed and unexpected gems are discovered.

Doing mathematics is a *creative* process. Only a few people can create really good music. Only a few people can create a really good book or movie. Similarly, only a few people can create really good math.

I believe that you can enjoy math more by becoming more comfortable and fluent with a small collection of simple ideas.

1.1 Brief Apology

Reading and then rereading what I've written here, it's clear that there are deficiencies in my writing style.

Something thing in particular that sticks out to me is the heavy and probably inconsistent use of parentheses, italics, and quotation marks.

Another things is that there are still some typos and broken sentences.

I hope to improve these things at some point in the future, and thanks for your patience in the meantime.

1.2 Why Did I Write This?

I want to make clear my intentions regarding you're reading.

I am attempting only to:

- Show you *a way of looking at things* which has been *helpful for me*.
- Provide you with *one particular starting point*, which *might be helpful to you* if you are new to reading math that someone else has written.
- Hopefully save you time, reduce frustration, and increase enjoyment.

I am *definitely not* attempting to:

- Tell you what math is and what math isn't.
- Tell you whether or not objects that mathematicians talk about are "real" or not.
- Tell you what's "true" and what's "false", or even if that has any meaning outside of what we're going to do here.
- Make *any claims at all* about what the *Law of the Excluded Middle* or *Law of Non-Contradiction* might mean in general, if anything at all.
- Tell you where the boundary between math and the "real world" might be, whether or not it even exists, or whether or not it even makes sense to talk about it.
- Provide in any way a rigorous foundation for math, in the spirit of `https://en.wikipedia.org/wiki/Foundations_of_mathematics`.
- Provide you with a reality, a truth, meaning, or an ontology.
- Present myself as an expert in the foundations of mathematics, logic, or philosophy. I am *not* such an expert. I am someone who is just pretty good at math. That is very distant from being an expert in foundations.

It turns out that a lot of very smart people have very different opinions on what mathematics and mathematical "truth" actually are. If that kind of thing interests you, then you are interested in the *philosophy of mathematics*.

See: https://en.wikipedia.org/wiki/Philosophy_of_mathematics

We seek to get on the same page as the authors that we wish to read. This means that we need to become familiar with some fundamental ideas. Often, complicated-looking things can be easily understood in terms of a relatively small collection of basic concepts.

In the end, though, maybe we shouldn't *always* try to define *every* concept in terms of *more fundamental* concepts. After all, how could you ever be satisfied that you've reached the bottom, that you've reached the *most fundamental* concepts? Even a five-year-old kid asking the question "but why...?" might shatter your convictions.

And so, I like these quotes from Kleene and Mendelson:

> The formally axiomatized propositions of mathematics cannot constitue the whole of mathematics; there must also be an intuitively understood mathematics. ... The metatheory belongs to intuitive and informal mathematics. ... The metatheory will be expressed in ordinary language.
>
> -Stephen Cole Kleene, *Introduction to Meta-Mathematics*

> Impeccable definitions have little value at the beginning of the study of a subject.
>
> -Elliott Mendelson, *Introduction to Mathematical Logic*

1.3 Coookies at the Grocery Store

Let us just quickly notice two ways of reasoning that we use commonly and effortlessly in daily life.

1.3.1 The first time you go to get cookies

You have been sent to the store to get cookies. You arrive at the store, and ask an employee where the cookies are. They reply

> "We definitely have cookies in the store right now, and they're either in aisle 1 or aisle 2, but I can't remember exactly which one".

The person who helped you is an expert about the store and they have no reason to lie. You believe what they say.

You walk over to to aisle 1, but there are no cookies there.
So, you then walk over to aisle 2 and get your cookies.

1.3.2 The second time you go to get cookies

You have been sent, again, to get cookies at the store.

You already know that if you need to get cookies, then you should go to aisle 2.
So, you walk straight over to aisle 2 and get your cookies.

1.4 Context

Context is the setting that frames new information. The meaning of words depends on *context*. This is not just a math thing - this is a feature or bug of human-to-human communication. For example, you are probably familiar with things called Homonyms.

A wedding *ring* is not the same thing as the *ring* of a bell. Without extra information, you might not know how to interpret the word "ring".

Context matters. What one mathematician calls an "algebra of functions" might be similar but not quite the same thing that another mathematician calls an "algebra of functions". Part of the *context* you have when reading is *who the author is*, and what they have provided you so far.

An important component of *context* is what types of things people are trying to communicate about. In some arenas, the totality of things to be taken into consideration in some context is sometimes called the *Universe of Discourse* or *Domain of Discourse*.

1.5 People communicate in writing by using symbols

This is because we can't actually insert most of the things we want to write about directly into a page! If we want to write about the planet Earth then we have to use a *symbol which refers* to it. And since we're forced to refer to *some* things by using symbols for them, the convention in writing has become to refer to *all* things by using symbols for them.

To determine if you have a decent understanding of how symbols and words are used *in writing*, consider the following eight sentences.

> The name of my dog Milo is "Milo".

> The name of the number immediately after 2 is "3".

> The statement "$\delta^2 + 10 = 3$" is false if $\delta = 4$.

> The notation "$\sqrt{(\cdot)}$" is employed to indicate the square root operation.

> The name of my dog Milo is Milo.

> The name of the number immediately after 2 is 3.

> The statement $\delta^2 + 10 = 3$ is false if $\delta = 4$.

> The notation $\sqrt{(\cdot)}$ is employed to indicate the square root operation.

If the first four of these seem correct and the last four seem questionable, then you're probably good to go.

To get more information, see the appedix section *"B Names, Symbols, and Notation"*.

Also related and important is the item *"Warning: Symbols are not functions!"* which appears after a brief discussion of what functions are.

1.5.1 What symbols and words mean depends on context

Lots of words in the English language have more than one meaning, and so the way you interpret them depends on *context*. Here are just a couple of examples:

- "You have a lot of *land* on your property."
  "The spaceship will *land* on Mars."
- "I like to *run*."
  "The Tigers didn't win the World Series this year, but they had a good *run*."

The situation in math is similar even though situations tend to be less ambiguous. How to interpret a name, word, or symbol which has been provided to you sometimes depends on context. The author's intentions, their own opinions, the information they expect you to already have – these are just three pieces of the context you have available when trying to interpret something they've provided you.

The point is this: You shouldn't expect to always be able to immediately interpret something someone blindly points to on some random page of a random book or article. Look at what's in the shaded box below:

$$(((x + y) + z) \cdot w)$$

How should you interpret the symbol "$(((x + y) + z) \cdot w)$" which appears in the above box? It depends on *context*.

- If the context is a primary school algebra class or a first-year university mechanical engineering class, then the symbol "$(((x + y) + z) \cdot w)$" might be interpreted as:

  > An informally-defined "algebraic expression" consisting of numbers, symbols and operators (such as "+" and "·") grouped together that show the value of some phisical quantity such as the weight of something.

- If the context is a proof-oriented analysis or algebra book for upper-level university students, then "$(((x + y) + z) \cdot w)$" might be interpreted as:

The value $F(x, y, z, w)$ of the function $F : \mathbb{C}^4 \to \mathbb{C}$ at a certain 4-tuple $(x, y, z, w) \in \mathbb{C}^4$, where F is defined as the composite function:

$$F = f_3 \circ f_2 \circ f_1$$

$$(\xi_1, \xi_2, \xi_3, \xi_4) \overset{f_1}{\longmapsto} (\xi_1 + \xi_2, \xi_3, \xi_4)$$

$$(\xi_1, \xi_2, \xi_3) \overset{f_2}{\longmapsto} (\xi_1 + \xi_2, \xi_3)$$

$$(\xi_1, \xi_2) \overset{f_3}{\longmapsto} (\xi_1 \cdot \xi_2)$$

- If the context is a book about mathematical logic or metamathematics, then "$(((x + y) + z) \cdot w)$" might be interpreted as:

  The string,
  finite characters sequence of length 13,
  13-tuple of characters, or
  character-valued function on the set of intengers from 1 to 13
  which is given by:

$$\begin{array}{ll} 1 \mapsto \underline{(} & 8 \mapsto \underline{+} \\ 2 \mapsto \underline{(} & 9 \mapsto \underline{z} \\ 3 \mapsto \underline{(} & 10 \mapsto \underline{)} \\ 4 \mapsto \underline{x} & 11 \mapsto \underline{\cdot} \\ 5 \mapsto \underline{+} & 12 \mapsto \underline{w} \\ 6 \mapsto \underline{y} & 13 \mapsto \underline{)} \\ 7 \mapsto \underline{)} & \end{array}$$

2 Elementary Concepts

Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.
(God created the integers, everything else is the work of man.)
-Leopold Kronecker, 1886, in speech to the Berliner Naturforscher-Versammlung

We cannot expect that the cognizance of the natural number sequence can be reduced to that of anything essentially more primitive than itself.
-Stephen Cole Kleene, *Introduction to Meta-Mathematics*

Kronecker and Kleene are getting at the idea that it might be quite difficult to actually *define* the natural ("counting") numbers in terms of other things which are is some important way *essentially more fundamental*. Kleen uses the word "primitive" in this context. They are not necessarily saying that you *can't* define the natural numbers in terms of other concepts. They are saying that if you do try, then these other concepts will likely be similiar enough to the natural numbers that you wind up gaining little, if anything at all.

See: `https://en.wikipedia.org/wiki/Primitive_notion`

The goal is to provide a decent starting point for understanding math. So, I'm going to list out the concepts which I tend to think of as primitive in the sense that Kleene used it above.

But people who are experts in the foundations of mathematics, logic, or philosophy tend to use the word "primitive" with a particular precise meaning. I am not an expert in the foundations of math, and I'm not trying to sound like one, so I shouldn't take the risk of using the word "primitive" in the context here.

Instead, I will use the word "*elementary*", and list out what I think of as *elementary concepts*. The actual word used doesn't really matter, but there's no need for unnecessary confusion.

Elementary concepts, ideas, or things will just be those concepts, ideas, or things which, for some reason or another, I just don't try to explain in terms of other concepts.

While it's not important for us here, it turns out that some of the items we will look at are in fact actually be considered *primitive* by the experts.

2.1 List of *Elementary Concepts*

Here is the list.

- **Simple counting and comparing that you could do in kindergarten**
  I think we need this here to even make sense of what you'll read further down. It appears difficult to even "reason about reasoning" without basic counting and comparing.

  Let's just agree that we know how to do simple counting and arithmetic using our fingers. We require:

  1. Understanding the numbers 0 up through 10 although it doesn't really matter how far we go.
  2. Especially understanding these: 0, 1, and 2.
  3. Understanding these in simple counting: more, greater, fewer, less.
  4. Understanding these in simple counting: at least (no/not fewer than), at most (no/not more than).
  5. Understanding these in simple counting: none, (exactly) one, at-least-one (some), all.
  6. Understanding basic addition in the context of simple counting. Or at least, what it means to get "one more of" or "add one to" something.

  Note: We will visit *All/Both/Universality* and *At-Least-One/Existentiality* later.

- **Process of Elimination**
  Quickly jump back to the section *"The first time you go to get cookies"*.
  What's going on is:

"I know there are two possibilities, and if the first one is wrong, then the second one has to be right."

More precisely:

- You know that, in some context, there are two possible scenarios, and that at least one of them must occur.
- You somehow determine that the first scenario does not or can not occur; in other words you can somehow "rule it out".
- Then you're forced to conclude that the second scenario *must in fact occur*.

Let's put the cookie example into this form:

- You know that there are two scenarios: cookies in aisle 1 or cookies in aisle 2. You know that at least one of them must occur.
- You determine that the scenario where cookies are in aisle 1 *does not* occur.
- You conclude that the the cookies *must in fact be* in aisle 2.

We may freely apply this way of reasoning in any situation, *except possibly* when reasoning "from within" an artificially created system wherein it has been disallowed (explicitly or implicitly).

Note: There is a difference between, on one hand, reasoning "within" a system as a participant bound by the system's logic, and on the other hand, reasoning "about" a system as an outside observer. It is like the difference between, on one hand, being a baseball player whose actions within the game are restricted by the rules, and on the other hand, being an outside observer of the game who is free to act without being restricted by the rules.

Some people might take issue with considering this mode of thinking as *elementary*, and perhaps protest with something like "No, you're just working with a truth table" (`https://en.wikipedia.org/wiki/Truth_table`). But, consider the thought process which is *occurring while trying* to draw conclusions from a truth table.

A fancy name for this kind of thinking is *modus tollendo ponens*.

See: `https://en.wikipedia.org/wiki/Disjunctive_syllogism`

- **Implication Elimination**
  Quickly jump back to the section *"The second time you go to get cookies"*.
  What's going on is:

  "I know thing #2 is true whenever thing #1 is true, and I see that thing #1 is true, so that means thing #2 also has to be true."

  More precisely:

- You know that scenario B *must* occur *if/when* scenario A occurs.
- You observe that scenario A *actually occurs*.
- And so, you conclude that scenario B *must in fact occur*.

Let's put the cookie example into this form:

- You know that scenario B (go to aisle 2) *must* occur *if/when* scenario A (need to get cookies) occurs.
- You need to get cookies (scenario A occurs).
- And so, you *must* go to aisle 2 (scenario B occurs).

Just as with *Process of Elimination*, I consider *Implication Elimination* to be *elementary*.

We may freely apply this way of reasoning in any situation, *except possibly* when reasoning "from within" an artificially created system wherein it has been disallowed (explicitly or implicitly).

A fancy name for this kind of thinking is *modus ponens*.

See: `https://en.wikipedia.org/wiki/Modus_ponens`

- **True, truth**
  Something is *true* when it has *truth*.
  For our purposes, these will mean the same thing ("[" and "]" added for clarity):

  - The apple is red.
  - The situation is that the apple is red.
  - It is *true* that the apple is red.
  - It is *true* that [the situation is that the apple is red].
  - The situation is such that [it is *true* that the apple is red].

  Important: The *true* and *false* we're talking about here should not be confused with the things called "truth values" which are defined in formal languages.

- **Not, other**
  These mean the same thing:

  - The apple is *not* red.
  - It is *not* the situation that the apple *is* red.
  - The situation is *other than* one where the apple *is* red.

  In general, these mean the same thing:

  - *thing1* is *not thing2*.

  - It is *not* the situation that *thing1 is thing2*.
  - The situation is *other than* one where *thing1 is thing2*.

- **False, falsity**
  False means exactly the same thing as *not true*.
  If something is *false*, then it has *falsity*.

  Important: The *true* and *false* we're talking about here should not be confused with the things called "truth values" which are defined in formal languages.

- **True vs false**
  Note: The word "statement" below is being used with a specific meaning which will be clarified a little further down. For the time being you can just think "a sentence stating a fact".

  The *Law of Excluded Middle* and the *Law of Non-Contradiction*, are not really "laws" at all, but instead provide guidelines for reasoning.

  - ***Law of Excluded Middle***:
    At least one of these situations occurs for a given statement:
    1. The statement is *true*.
    2. The statement is *not true*.
  - ***Law of Non-Contradiction***:
    Not more than one of these situations can occur for a given statement:
    1. The statement is *true*.
    2. The statement is *not true*.

  Taken together, these capture the specific relationship between *true* and *false* which is the particular one *most commonly* used in mathematics.

  Here's a rule of thumb.

Applying *Law of Excluded Middle* and *Law of Non-Contradiction*

> We may freely apply *Law of Excluded Middle* or *Law of Non-Contradiction* in any situation, *except possibly* when reasoning "from within" an artificially created system where one or both have been disallowed (explicitly or implicitly).
>
> > *Note: Performing symbolic manipulations while working within the confines of a formal language, which has a list of rules specifying the admissible manipulations, is a good example of reasoning "within" an artificially created system. Specifically, in an automated theorem proving application, the Law of Excluded Middle either would or would not be included as a rule the program can apply, depending on the needs of the programmer.*

> A scenario which violates *Law of Excluded Middle* or *Law of Non-Contradiction* cannot occur, unless you are working "inside of" such an artificially created system.
>
> The upshot is that you can almost always apply *Law of Excluded Middle* and *Law of Non-Contradiction* when you are attempting to follow a mathematician's thought process.

This means that we may reason as follows.

A way of ruling out scenarios when using *Process of Elimination*

> If you are reasoning using *Process of Elimination*, and you discover that one of your possible scenarios violates either the *Law of Excluded Middle* or the *Law of Non-Contradiction*, then that scenario *can not occur*, and so can be can be "ruled out".

As an application, let's convince ourselves of this:

- **Combined Law of Excluded Middle and Non-Contradiction**:
  One and only one of these situations can occur for a given statement:
  1. The statement is *true*.
  2. The statement is *not true*.

  In other words: A statement is either *true* or *false (not true)*, but not both.

  Justification:

  > You can reach this *"Combined"* conclusion by using *Process of Elimination*. The three possible scenarios to examine are:
  > 1. Scenario 1: Neither situation occurs.
  >   This violates *Excluded Middle* and so cannot occur.
  > 2. Scenario 2: Both situations occur.
  >   This violates *Non-Contradiction* and so cannot occur.
  > 3. Scenario 3: Exactly one situation occurs.
  >   This *must occur* since the other two have been eliminated.

As another application, let's show that:

- **Cancelling double negatives**:
  "Double negatives" cancel each other out:
  A statement is *not [not true]* exactly when it is *true*.
  A statement is *not [not false]* exactly when it is *false*.

  Justification:

Let's say you have a statement A.

What does it mean for A to be *not [not true]*?
It means:
It is *not* the situation that [A is *not true*]
But then by *Combined Law of Excluded Middle and Non-Contradiction*:
It must be the situation that [A is *true*].
In other words:
A must be *true*.
So, if A is *not [not true]*, then we can conclude that A is *true*.

On the other hand, what does it mean for A to be *true*?
It means:
It is the situation that [A is *true*]
But then by *Combined Law of Excluded Middle and Non-Contradiction*:
It is *not* be the situation that [A is *not true*].
In other words:
A must be *not [not true]*.
So, if A to be *true*, then we can conclude that A is *not [not true]*.

We just figured out that something is *not [not true]* exactly when it is *true*.

What about *not [not false]*?
Well, *false* is the same as *not true*.
So, *not [not false]* is therefore the same as *not [not (not true)]*
... which is the same as *not [true]*
... which is just *not true*
... which is the same as *false*.

See these:
`https://en.wikipedia.org/wiki/Law_of_noncontradiction`
`https://plato.stanford.edu/entries/contradiction/`
`https://en.wikipedia.org/wiki/Law_of_excluded_middle`
`https://plato.stanford.edu/entries/contradiction/`
`https://en.wikipedia.org/wiki/Intuitionism#Truth_and_proof`
`https://en.wikipedia.org/wiki/Philosophy_of_mathematics`

Note: There is an very interesting connection between the *Law of Excluded Middle* and the so-called *Axiom of Choice*. Google it.

- **What does it mean in math for something to be *meaningful*?**
This is a general idea which also applies to math/cs/logic/philosophy.

Consider these two sentences:

1. *My dog is brown.*
2. *This sentence is false.*

Most people would say that the first sentence is a fairly coherent statement which "make sense" and is in some way *meaningful*, even if they don't necessarily agree that the dog is brown. The intended meaning of the first sentence is clear.

On the other hand, a lot of people would be confused by the second sentence, and might be inclined to say it *doesn't* "make sense", or isn't *meaningful*. Wrestling with it in order to determine the intended meaning results in nothing but confusion.

Fortunately, this kind of confusion which the second sentence causes isn't usually a problem when you are reading math. Typically you are reading something written by someone who knows what they're doing, and has done a good job of arranging things so that they in fact actually do "make sense".

Can we nail down more precisely what *meaningful* means in mathematics?

It turns out to be extremely difficult to nail down precisely without working inside a formal language (finite sequences of characters), and we are absolutely not working inside a formal language right now. Indeed, what we're doing here – in what you're reading – is trying to get a handle on the situation when *reasoning informally within informal mathematics (and this includes reasoning about abstract pure math!)*, or when *reasoning informally about a formal language as an observer* outside and above that formal language. (Reasoning informally about a formal language is similar to using English to describe German grammar.)

So it's tough to say exactly when something's *meaningful*.
Because of this, it will have to be good enough for our purposes to give a couple of situations where there might be problems.

One reasonable requirement for something to be *meaningful* is that all component parts of whatever's in question should probably themselves be *meaningful* in the *context* of whatever's in question. For example, many people would say that – in the current context you're reading right now – the following would not be *meaningful*:

> ASDJHGJASHDGKJASD is awesome.

The sequence of letters "ASDJHGJASHDGKJASD" has no meaning, so the above attempt at using it to form a sentence is meaningless.
Similarly, many people would acknowledge that

> The Jolly Green Giant living on the moon is awesome.

is a correctly written sentence which is nevertheless not *meaningful*, because it is making a statement about something (Jolly Green Giant living on the moon) which does not exists.
This is to be contrasted with the following:

> If the Jolly Green Giant living on the moon exists, then your name is "ASDJHGJASHDGKJASD".

The utterance

> This statement is false

is another example of something almost universally considered to be not *meaningful*. One common view of this situation is that, roughly speaking, something is *meaningful* when it's not "unacceptably circular or self-referential". A word that is often used to describe something which fails to be meaningful in this way is *impredicative*.

This rough way of describing *meaningful* is just that: rough. The benefit is the acknowledgment that circular or self-referential sentences or expressions can be problematic. But what separates "acceptably self-referential" from "unacceptably self-referential"?

It turns out to be very difficult to nail things down with complete precision. In fact, many great mathematicians, logicians, and philosphers have wrestled with this profound question. Among them are Poincaré, Russell, Whitehead, Cantor, Weyl, Zermelo, Ramsey, and Gödel. It is still actively argued.

Short story: If someone throws a paradox at you and you want to seem like you know what you're doing, just say something like "I think there's an *impredicative* situation here, so the whole thing is likely meaningless, and if you could please go away..."

See:
`https://en.wikipedia.org/wiki/Impredicativity`
`https://www.iep.utm.edu/predicat/`
`https://puzzling.stackexchange.com/questions/6653/`

Feel free to skip the rest of this item, but if you're interested...

> A statement or definition is *impredicative* if it is self-referential in certain specific ways. It is not always immediately obvious when you're in such a situation! *Sometimes* when something is impredicative, it winds up also being unacceptably circular, not meaningful, paradoxical. But *other times*, things wind up being just fine.
>
> In fact, it seems that there is not yet a complete consensus about when an impredicative situation is legal, and when it's illegal.

You already saw the Liar's Paradox: "This sentence is false". This is impredicative. It is transparently self-referential, and is also unanimously considered to be an unacceptable impredicative situation.

Russel's Paradox provides another impredicative situation which is also considered unacceptable. But, exactly *why* it's self-referencing is harder to see. Here are two equivalent ways of trying to define the *Russell set*:

The *Russell set* is the set R ...

1. ... consisting of of all sets which are not members of themselves.
2. (or equivalently) ... which satisfies: For *every* set s, $s \in R$ if and only if $s \notin s$.

This situation is self-referential because the definition involves taking into consideration (via "quantifying over") a totality (namely the class of all sets) which *includes the thing we are trying to define.*

To see the paradox related to this self-referential definition, consider whether or not R is or is not a member of itself.

Here are some examples of impredicative definitions which most people do not consider troublesome:

- Define *Julius* to be the tallest person in the room.
- Given a non-empty set X of real numbers, define the *least upper bound* of X to be the upper bound of X which is less than or equal to every other upper bound of X.
- Define the property N (of being a natural number) as: An object x has the property N if and only if x has every property F which is had by zero and is inherited from any number u to its successor $u + 1$.

Some paradoxes to explore if you are interested:

- Burali-Forti Paradox
- Russell's Paradox
- Lairs's Paradox
- Berry's Paradox
- Löb's Paradox
- This paradox which has turned into an internet meme:

  If you choose an answer to this question at random, what is the chance you will be correct?
  A) 25%
  B) 50%
  C) 60%
  D) 25%

- **Property**
  Roughly, a *property* is something *specific*, *unambiguous*, and *meaningful*, that "can be said about" something.

  I'm going to weasel my way out of trouble by saying this: whether or not something is *specific*, *unambiguous*, and *meaningful* often *depends on the context*.

  Something either *does* or *does not* have a given property.
  It is either *true* or *false* that the thing has the property.

  When you say "The apple is red", then you are saying the apple has the *property* of "being red", or "redness".

  Examples:

  - The apple *is red*. Property: Being red.
  - I *am my father's son*. Property: Being one's father's son.
  - The number 9 *is the square of some integer*. Property: Being the square of some integer.
  - The number 9 *has an integer sqaure root*. Property: Having an integer sqaure root.

  You can *negate* properties to get new properties, usually by applying "*not*":

  - For example, the number 5 has the property that it *is the largest number* in the set $\{1, 2, 3, 4, 5\}$.
  - But the number 4 has the property that it *is NOT the largest number* in the set $\{1, 2, 3, 4, 5\}$.

  People often use uppercase letters such as P, Q, or R as shorthand for properties being talked about. For example, "Let R be the property of being red", or "Let P be the property of being a prime number".

  If P is shorthand for the property

  > *being the largest number* in the set $\{1, 2, 3, 4, 5\}$,

  then "(*not* P)" or "*not* P" is shorthand for the property obtained by *negation*:

  > *is NOT the largest number* in the set $\{1, 2, 3, 4, 5\}$.

  You can generally also combine properties in reasonable ways to get new properties.

  For example, if P and Q are properties, you can talk about a property R which is *true* if *both* of P and Q are *true* .
  You could also talk about a property R which is *true* if *at least one* of P or Q is *true*.

Example:

> Let P be the property of being a human, and Q be the property of weighing at least 1000 lbs.
> You can talk about a property R which is *true* if at least one of P or Q is *true*.
> The property R would be *true* for your human friend, but also for Mt. Everest.
> R would not be *true* for a very small rock which weighs 1 gram.

There is a shorthand which people use to make things faster to write.

If Q is a property, and b refers to something (like an apple or a car or whatever), then

> "Qb" or "$Q(b)$" is shorthand for "Thing b has property Q".

For example, if R is the property of being red, and c is my car, then

> "Rc" is shorthand for "My car is red".

See: `https://en.wikipedia.org/wiki/Property_(philosophy)`

- **Statement**
  Roughly, a *statement* is a sentence or idea which:

  1. asserts a truth or makes some claim,
  2. is unambiguous and meaningful,
  3. someone else could respond to with "Yes, that's true" or "No, that's not true",
  4. you could add after the words "Is it true that..." to make a meaningful question.

  Again, I'm going to weasel my way out of trouble by saying this: whether or not something is *specific*, *unambiguous*, and *meaningful* often *depends on the context*.

  A statement is either *true* or *false*.
  Most people would consider these to be statements:

  1. "The apple which you have in your hand is red".
  2. "*Every* apple is red".
  3. "*At least one* apple is red".
  4. "The apple is neon green and also knows how to build spaceships". Most people would say that this statement it is false.
  5. "I am my father's son."
  6. "The number 9 has an integer sqaure root."
  7. In general (see #1): "{*Specific thing*} has {*specific property*}"

8. In general (see #2): "{*Every thing*} has {*specific property*}"
9. In general (see #3): "{*At least one thing*} has {*specific property*}"

The above list is not supposed to be exhaustive.

These are not statements:

1. "This sentence is false."
   It is not meaningful.
2. "Let's go to the store."
   This sentence does not make a claim or assert a truth.
3. "Why is the sky blue?"
   This sentence does (directly) not make a claim or assert a truth.

Statements are sometimes called *sentences* or *propositions*.

- **Predicate**
  In math you will come across things which *look like* but *aren't quite* statements. The difference is entirely due to *context*, and which names or symbols *in that context* have or have not yet been assigned meanings.

1. "x is red",
   where x is some *placeholder* which does not yet refer to a specific thing.
2. "n is greater than 3",
   where n is some *placeholder* which does not yet refer to a specific thing.
3. In general:
   "{*placeholder*} has {*specific property*}",
   where the *placeholder* does not yet have any meaning.

   For example, with z as the *placeholder*:
   "z has property P", where z is a *placeholder* and P is a specific property.

   Shorthand:
   "Pz" or "$P(z)$" is shorthand for "z has property P".

The *placeholder* doesn't have meaning, so these items do not, as they stand, assert any truth or make any claim, and are called *predicates*. Only once the ambiguity of the *placeholder* is *somehow* removed is a *statement* created:

1. Let x refer to my car.

   "x is red"

   This is a *statement* asserting that something specific (my car) has some property.
2. Let $n = 2$ (n refers to the number 2).

"n is greater than 3"

This is a *statement* asserting (incorrectly) that something specific (the number 2) has some property.

3. "$m < m + 1$ for *every* number m".
   This is a *statement* which follows the pattern

   {*Every thing*} has {*specific property*}

   or in other words

   For {*every thing*} m, it is true that m has {*specific property*}

   In general, for a specific property P and placeholder m:

   "*Every thing* m has property P"

   or

   "Pm is true for *every* m"

4. "y is red for *at least one* automobile y".
   This is a *statement* which follows the pattern

   {*At least one thing*} has {*specific property*}

   or in other words

   There is {*at least one thing*} y such that y has {*specific property*}

   In general, for a specific property Q and placeholder y:

   "*At least one thing* y has property Q",

   or

   "Qy is true for *at least one* y"

- **Universality: all, always, every, each, both, and**
  The best thing to do here is to just give examples.

  - *All* dogs are mammals.
  - Dogs are *always* mammals.
  - *Every* dog is a mammal.
  - *Each* dog is a mammal.
  - For *every* thing x, if x is a dog, then x is a mammal.
  - For *everything* y, if y is a dog, then y is a mammal.
  - For *all* w, if w is a dog, then w is a mammal.
  - *Both* of these two dogs in front of me are mammals.
  - My dog is a mammal *and* your dog is a mammal *and* her dog is a mammal.
  - The following is *always* true for things x: If x is a natural number, then $x + 1$ is also a natural number.

- The following is true for *every* x: If x is a natural number, then $x + 1$ is also a natural number.
- *All* things are the same as themselves (whatever the heck that means).
- *Every* thing z satisfies $z = z$ (whatever the heck that means).
- For *each* n, if n is an even natural number, then $n + 1$ is an odd natural number.

Statements such as these are called *universal statements*.

Important! The word "any" is not invloved in the above. "Any" is ambiguous, and can be *either universal or existential depending on context*.

See: `https://en.wikipedia.org/wiki/Quantifier_(logic)`

- **Applying/using universality**
  What can you do with the knowledge that something is *always* true?

  Easy! You can *always* apply that information, in *every* single situation you can possibly encounter or imagine.

  Suppose I know that *every* time something is an apple, it turns out to also be a fruit. Say you now give me something in a box, and I can't see what's in the box. You tell me that the thing inside the box is an apple. If I believe you, then even without seing inside the box, I now also believe that there is a fruit in the box.

  In a nutshell, if something is *always* true, then it is true in *every specific* situation.

  Here's how this works in a more mathy context.
  Let P be some *property*.

  Suppose I believe that predicate Pw is *always* true ("*every* thing w has property P"). I conclude that P then holds for any specific thing which I might be looking at. In other words, it doesn't matter *what* you substitute for w, you still conclude that this specific thing has property P: if w is a dog, then this dog has property P, if w is an apple then the apple has property P, if w is the square root of 42 then it has property P, if w is an alien from an alternate universe which you are only imagining then the alien has property P, and so on.

  You can, *every time and without exception*, conclude that property P holds.
  Property P is *universal*.

  Let's go back to the original example with the apples and the fruits, an encode it using properties and predicates.

Let P be the property of a thing that it either *is not* an apple, or *is* a fruit.
In other words, for P to be *true* for a thing, at least one of the following must be *true*:

1. The thing *is not* an apple
2. The thing *is* a fruit

You can convince yourself that "Px is true for *every* x" says the same thing as "*every* apple is also a fruit".

- **Demonstrating/showing/proving universality**
  How do you demsonstrate to yourself or someone else that some claim or statement, or property, is *always* true?

  This is a very interesting idea which I wrestled with (but didn't know I was wrestling with) for a long time.

  Let's look at what it would mean to "prove" these (equivalent) statements:

  - For *each* n, if n is an even natural number, then $n+1$ is an odd natural number.
  - For *all* n, if n is an even natural number, then $n+1$ is an odd natural number.
  - For *every* n, if n is an even natural number, then $n+1$ is an odd natural number.
  - It is *always* true that $n+1$ is an odd natural number if n is an even natural number.
  - If n is an even natural number, then $n+1$ is *always* an odd natural number.

  This "proof" doesn't work:

  > "14 is an even natural number, and 15 is an odd natural number."

  Because, it gives *only one lonely example* of a situation where things work out the right way.

  If you were looking only at natural numbers less than, say, 5, then this kind of "listing out the examples" is basically good enough:

  - 0 is even, and 1 is odd.
  - 2 is even, and 3 is odd.
  - 4 is even, and 5 is odd.

  This kind of demonstration works if the "universe" of things you need to look at is finite.
  But, if you're making a claim about *all* natual numbers, of which there are infinitely many, then you can't just list out all the of examples.

  So, what can you do?

Well, there's hope. Mathematicians seem to in general agree that the preferred way to convincingly "prove" universality over an *infinite collection* of things is as follows:

1. "Choose/pick/grab/imagine/pluck-out-of-thin-air" an "arbitrary" thing from "anywhere" in any universe, real or imagined, possible or impossible.
   You may allow yourself to imagine reaching your hand up and plucking something out of thin air.
2. You have no information at all about this thing.
   You are not allowed to assume anything at all about it.
3. If you need to give the thing a name in order to work with it, you *must* choose a name which *has not yet been used* in whatever you're doing.
   This is in order to prevent accidentally and incorrectly using extra information about the thing, that you don't actually have.
   You don't want to confuse two different things by using the same name for both of them.
   Note: I'm using the word "name" very loosely. I just mean any way of *identifying* or *getting an handle on* a thing: which hand you're holding it in, which symbol you're using, where in your field of vision you're imagining it's located, etc.
4. Demonstrate that this thing has the desired property.

That's it. The upshot is, *universality of statements is demonstrated by showing truth of statements in the total absence of information.*

Let's consider again the following claim:

- For every n, if n is an even natural number, then $n + 1$ is an odd natural number.

The following three fake arguments *do not* correctly demonstrate the truth of the claim.

Fake argument #1 - Treating an infinite collection like a finite one:

0 is even, and 1 is odd.
2 is even, and 3 is odd.
4 is even, and 5 is odd.
...etc...
Proceeding in this way we see that the claim is true for all natural numbers.
Done.

Fake argument #2 - Using extra information, or using a specific example:

Let n be some even natural number, for example say $n = 10$.
Since $n = 10$, we know $n + 1 = 10 + 1 = 11$.
11 is odd.
Done.

Fake argument #3 - Not using a new/fresh name for the arbitrary chosen thing:

Let $m = 24$.
Now, pick an arbitrary thing out of thin air, and call it m.
If $m = 24$, then $m + 1 = 25$.
m is an even natural number, and $m + 1$ is an odd one.
Done.

Here is a solid, correct, and generally accepted argument which demonstrates the claim:

Solid, correct, and generally accepted argument:

(Assuming here we understand "even" and "odd".)

Pick an arbitrary natural number, and call it m.
We need to show that if m is even, then $m + 1$ is odd.
Suppose that m is even.
This means that $m = 2k$, for some natural number k, by definition of "even".
Then $m + 1 = 2k + 1$.
But $2k + 1$ is odd, by definition of "odd".
Done.

Important! There is actually another way to prove *universality*, by using duality with *existentiality*. See the section about that duality.

- **Existentiality: or, some, sometimes, at least one, exists, there is, there exists**
  Like with *universality*, maybe the best thing I can do to describe *existentiality* is just to give examples.
  - *At least one* of the dogs in the show is a Vizsla.
  - Dogs are *sometimes* Vizslas.
  - Humans are *sometimes* female.
  - *Some* human is a female.
  - *There exists* a human which is a female.
  - *There is* a human which is a female.
  - This dog *or* that dog is a white dog.
  - The first dog *or* the second dog *or* the third dog is a brown dog.
  - A brown dog *exists*.
  - *There exists* an x such that the following holds: x is a natural number and $x^2 = 9$.
  - The following is true for *some* x: x is a natural number and $x^2 = 9$.
  - *There is some* x such that the following holds: x is a natural number and $x^2 = 9$.

  Statements such as these are called *existential statements*.

Important! The word "any" is not invloved in the above. "Any" is ambiguous, and can be *either* universal or existential *depending on context*.

See: `https://en.wikipedia.org/wiki/Quantifier_(logic)`

- **Applying/using existentiality**
  Consider a property P, for example, maybe P is the propoerty of being a Vizla dog.
  If you know that *there exists* something which has property P, then you are allowed to reason like this:

  1. You may "choose/pick/grab/imagine/pluck-out-of-thin-air" a thing which is an *example* of the things which have property P.
  2. You have no information about this thing at all, and may not assume anything about it, other than that it has property P.
  3. If you need to give the thing a name in order to work with it, you *must* choose a name which *has not yet been used* in whatever you're doing.
     This is in order to prevent accidentally and incorrectly using extra information about the thing, that you don't actually have.
     You don't want to confuse two different things by using the same name for both of them.
     Note: I'm using the word "name" very loosely. I just mean any way of *identifying* or *getting an handle on* a thing: which hand you're holding it in, which symbol you're using, where in your field of vision you're imagining it's located, etc.
  4. You may now work with this thing and use it for whatever purpose you want.

  Here is an example of how to correctly use existentiality.

  > Fact #1: You know that *there exists* a thing which is a Vizla dog (so it is a dog and more specifically is a Vizla).
  > Fact #2: You also know that *every* dog is also a mammal.
  >
  > The first fact (existentiality) allows you to "choose" a thing which is a Vizla dog.
  > Call this thing "z", which is a brand new and unused symbol in this context.
  > Now you have a thing z which is a Vizla dog.
  > The second fact (universality) allows you, as described earlier, to conclude that z is a mammal.
  > You are now working with a thing you have called "z", and which you know is a mammal.
  > You can go on and use it for whatever purpose you need to use it for.

  Here is an example of using existentiality incorrectly.

  > Let's incorrectly try to "prove" that *all* mammals are Vizla dogs.
  > We are trying to prove a statement of *universality*.

Fact: You know that *there exists* a thing which is a Vizla dog.

From the section about universality, there is a four-step process for proving universality.
Steps 1, 2, & 3: Pick an arbitrary mammal, call it m.
Step 4: Show that m is a Vizla dog. We know there exists a Vizla dog, so pick one and call it m. Thus our arbitrary mammal m is a Vizla dog.

Obviously the mistake is that we incorrectly confused our example of a mammal and our example of a Vizla dog. This happened because we incorrectly used the name "m" for two possibly different thinigs. This is why you can't reuse names/symbols in arguments like this.

Although most of us don't usually make this kind of mistake on a day-to-day basis, it's still important to have it officially codified in order to avoid making mistakes when you're distracted by working on more complicated things.

- **Demonstrating/showing/proving existentiality**
  The idea is basically simple.

  To demonstrate *existentiality* for some property, all you have to do is some provide or come up with an example of a thing which has that property. If I want to demonstrate the truth of the statement "*There exists* an x such that Px is true", then all I have to do is somehow produce a thing which has property P. Obviously this is sometimes easier said than done.

  Here's an example.

  Let S be the property of being a number which is the product of three distinct prime numbers.
  You want to show that Sn is true for *some* n.

  Let's look at the number $t = 105$.
  $105 = 3 \cdot 5 \cdot 7$ and so St is true (t is *specific thing* here, not a *placeholder*).

  So you just produced an example, namely $t = 105$, of a thing which has property S. Since you found such an example, you may now conclude that Sn is true for *some* n.

  In the example above, we didn't really need to use the name/symbol "t" to get a handle on the number 105. But in more complicated situations, it's usually helpful and sometimes necessary to give things names. It helps keep everything organized and manageable.

Important! There is actually another way to prove *existentiality*, by using duality with *universality*. See the section about that duality.

- **Short-circuiting and/or, universal/existential**
Here is some potential trouble which will pop up fairly frequently in math and computer programming.

    Imagine, for simplicity and just for the moment, that it doesn't make sense to talk about square roots of negative numbers. In other words, pretend for the moment that you don't know about complex numbers.

    Let $n = 9$, and consider these four statements:

    1. $\sqrt{n}$ equals 3 *and* n is greater than 0.
    2. $\sqrt{n}$ equals 3 *or* n is less than 0.
    3. $\sqrt{n}$ equals 3 *and* n is less than 0.
    4. $\sqrt{n}$ equals 3 *or* n is greater than 0.

    All of this group of four sentences are meaningful statements. There is no trouble. The first is *true* since both parts are *true*. The second is also *true*, since at least one of the parts is *true*. The third is *false*, becasue not both parts are *true*. The fourth is *true*, since both parts are *true*.

    But now let $n = -2$, and consider the next four "statements":

    1. $\sqrt{n}$ equals 3 *and* n is greater than 0.
    2. $\sqrt{n}$ equals 3 *or* n is less than 0.
    3. $\sqrt{n}$ equals 3 *and* n is less than 0.
    4. $\sqrt{n}$ equals 3 *or* n is greater than 0.

    Here be dragons! We might be in trouble. We don't know how to make sense of $\sqrt{-2}$. It is *meaningless* or *undefined* in our context here, and so each of the sentences contains something which is *meaningless/undefined*.

    So this is the question: Which, if any, of this second group of four items are generally considered *meaningful statements*, despite the fact that each has a *meaningless* component?

    The answer is that, by convention, two of the four are assigned *true/false*, and the other two are decreed to be *meaningless*:

    1. *False*, by convention.
    2. *True*, by convention.
    3. *Meaningless/undefined*, by convention.
    4. *Meaningless/undefined*, by convention.

Short-circuiting is the name for this convention, which is nearly universally applied in such situations, typically without being explicitly mentioned. It allows you to find meaning in situations where the *meaningless* components in a sense "do not affect the outcome".

Here's general idea, roughly:

1. Suppose you have a "statement-like" sentence, which has some *mening-less/undefined* components.
2. Imagine what would happen if the meaningless components were individually, one by one, and independently, replaced by *true* or *false*.
3. Suppose you always wind up with a genuine *statement* when you do this.
4. Suppose in addition that the resulting statement is *always true* or *always false*, no matter how you make sense of the meaningless components as *true* or *false*.
   In other words, whether the resulting statement is *true* or *false* "does not depend" on the meaningless components.
5. In this case, the *short-circuiting convention* is that your "statement-like" sentence can be considered a genuine *statement*, which is considered to be *true* or *false*, according to what you found above.

Here are applications in a few specific scenarios.

1. ***A and B.***
   If one of A or B is *false* and the other is *meaningless*, then the whole statement

   A and B

   is generally considered to be *false*. It doesn't matter if the meaningless component is replaced by *true* or *false*, since the resulting statement is *false* in each case.
2. ***A or B.***
   If one of A or B is *true* and the other is *meaningless*, then the whole statement

   A or B

   is generally considered to be *true*. It doesn't matter if the meaningless component is replaced by *true* or *false*, since the resulting statement is *true* in each case.
3. **Property Q is *true* for *at least one* thing.**
   If Q is some *property* which is known to be *true* for *some specific* thing, but which might be *meaningless* or *undefined* for other things, then the whole statement

   Qy is *true* for *at least one* thing y

   is generally considered to be *true*.

   In other words, you can say that Q is *true* for *at least one* thing, even if it doesn't always make sense to talk about Q, so long as there's an example where Q is both *meaningful* and *true*.
4. **Property Q is *false* for *at least one* thing.**
   If Q is some *property* which is known to be *false* for *some specific* thing, but which might be *meaningless* or *undefined* for other things, then the whole statement

Qy is *false* for *at least one* thing y

is generally considered to be *true*.

5. **Property P is *true* for *every* thing.**
If P is some *property* which is known to be *false* for *at least one* thing, but which might be *meaningless* or *undefined* for other things, then the whole statement

Px is *true* for *every* thing x

is generally considered to be *false*.

- **Always/sometimes duality, and/or duality**
There is an absolutely beautiful duality between *always* and *sometimes*, often referred to as "De Morgan's Law", "De Morgan's Rule", or "De Morgan Duality". Anybody who's spent signigficant time computer programming has probably applied De Morgan Duality, maybe without knowing there's a name for it. It appears in programming when you're trying to simplify conditional "if/else/endif" structures.

> Important!
> The duality we're about to look at isn't exactly *elementary* in the sense we've been using the word. It is in fact *provable* using only stuff about *universality* and *existentiality* that we've already talked about. There is a proof in a later section.

Here's the basic idea:

> Let's say you have 100 gumdrops, and they're all either licorice or spearmint. Say you hate licorce gumdrops and you want to find any so you can foist them upon some unsuspecting victim. You examine all 100 gumdrops one by one.
>
> You either find a licorice one or you don't.
>
> <u>If you find a licorice one</u>
> *At least one* of them is licorice.
> It is *not true* that they are *all* spearmint.
>
> <u>If you do not find a licorice one</u>
> It is *not true* that *at least one* of them is licorice.
> They are *all* spearmint.
>
> More succinctly, you can say:
> *At least one* gumdrop is licorice *exactly when* it's *not true* that *every* gumdrop is spearmint.

The gumdrops example entirely captures the spirit behind De Morgan Duality.

Now let's move on to the general versions.

Version # 1: *Universality/Existentiality Duality*

Let P be some property, and let (*not* P) be the property obtained by *negation* (we talked about this earlier...)

At least one thing has property (*not* P)
if and only if
it is *not* the situation that *every* thing has property P.
Example:
At least one thing is *not red*
if and only if
it is *not* the situation that *every* thing is *red*.

Version # 2: *AND/OR Duality*

Suppose you have two statements.

At least one of the statments is true if and only if it's *not* the case that they are *both* false.

Said another way, for statements A and B:

(A *or* B) is *true* if and only if ((not A) *and* (not B)) is *false*.

This appears in programming as:

$(a \parallel b)$ == !(!a & !b)

- **Impossible, never true, can't happen, can't be true**
  In math, "*impossible*" just means *false* or *always false*, depending on the context.

  General guideline:

  - If Q is a property, "Q is *impossible*" means:
    Q is *always false*.
    Rephrased:
    Qw is *false* for *every* thing w.

  - If B is a statement, "B is *impossible*" means:
    B is *false*.

- **Implication, implies, guarantee, guarantees, promise, promises**
  Let's talk about what it means in math for one thing to *guarantee*, or *imply*, another thing.

  Let A and B be statements.

> "*A implies/guarantees B*"
> is a statement which is *true* exactly when *A* is *false* or *B* is *true*, or both.

In other words:

> *A implies/guarantees B* is *true* exactly when you are in one of these three situations:
>
> 1. *A* is true and *B* is true
> 2. *A* is false and *B* is true
> 3. *A* is false and *B* is false

Rephrased yet again:

> *A implies B*
> is *true* precisely when it's *impossible* to have *A true* and *B false*.

Important! You should at all costs avoid the temptation of thinking about about *implication* in terms of "cause and effect". That's not what's going on here. The idea of *implication* that is used in math is about what combinations of *true* and *false* are possible between the *antecedent* *A* and the *consequent* *B*.

A guarantees B can be thought of as "*A* promises *B*": the implication is *true* except in the "broken promise" situation where *A* is *true* and *B* is *false*.

Example:

> Pick your favorite integer, and let *my_number* refer to that specific integer.
> Let *A* the statement
>
> $my\_number > 10$
>
> and let *B* the statement
>
> $my\_number > 5$
>
> Then the statement
>
> *A guarantees B*
>
> means that either *my_number* is not more than 10, or *my_number* is more than 5, or both. In other words, it is impossible for *my_number* to be both greater than 10 and less than 5.

Here is a detail which might seem strange at first:

> If the *antecedent* *A* is *false*, then the implication "*A implies B*" is *true*, no matter whether the *consequent* *B* is *true* or *false*.
>
> So for example, this implication is *true*:

> [Bryan climbs the Eiffel Tower every day] *implies* [we are living in the year 1969].

The implication is *true*, because it is *not true* that I climb the Eiffel Tower every day. No promise has been broken here.

See: `https://en.wikipedia.org/wiki/Material_conditional`

- **Thing, object, entity, item**
  For the purpose of reading math, a *thing/object/entity/item* is basically anything that you might refer to using a noun, in the broadest sense possible:

  – Something known in some sense to exist.
  – Something which possibly does not exist.
  – Something known in some sense to certainly not exist.
  – A concrete and tangible thing.
  – An idea or concept.
  – Etc.

  Important examples:

  – A wolf is a thing.
  – A pack of five wolves is a thing.
  – A *set/collection* (see below) is also a *thing*.
  – The number 42 is a thing.
  – The integer which is more than 0 but less than 1 is a thing, though it doesn't actually exist.
  – The sentence you are currently reading is a thing.
  – The *property* of being red, "redness", is a thing.
  (See the note below.)

  Note:

  > I am using *thing/object* in a very general way. There isn't any harm in doing so for the purposes of reading and understanding math. However, you should be aware that philosophers will sometimes use the word "*object*" in a more restricted sense allowing that there are "*non-objects*". In particular *properties* are sometimes considered to something other than *objects*.
  >
  > We don't need to worry about this different perspective, but if it interests you, then see:
  > `https://plato.stanford.edu/entries/object`
  > `https://en.wikipedia.org/wiki/Ontology#Some_fundamental_questions`

- **Same, equal, identical, indistinguishable**
  What it means in math for two potentially different things to actually *be the same thing* is *elementary* and truly fundamental.

  > It depends upon what the meaning of the word 'is' is.
  >
  > -President Bill Clinton

  Two potentially different *things* are in fact *the same/equal/identical/indistinguishable* if and only if every *property* possessed by one is possessed by the other.

  For two potentially different things referred to by a and b, the shorthand "$a = b$" means that a and b are actually *equal/identical*.

  Here's an example showing why this view of *equality* could be considered a natural one.

  > Let a refer to my Mazda car.
  > In other words, thing a *is* my Mazda.
  >
  > Suppose you know that b refers to *the same thing* as a does.
  > Suppose you also know that my Mazda is red.
  > This just means that thing a possesses the property of redness.
  > Since b refers to the *same* thing that a refers to, namely my Mazda, it is natural to conclude that thing b also possesses the property of redness.
  >
  > But now, on the other hand, suppose you know only that thing b possesses every property that thing a possesses.
  > Notice that "being my Mazda" is a property, which thing a in fact possesses!
  > Then thing b also has this property of "being my Mazda".
  > It is natural to conclude that thing b *is* my Mazda.

  It is really interesting that from this perspective on *equality* we can obtain, as a consequence, the Reflexive, Symmetric, and Transitive properties!

  <u>Equality is "Reflexive"</u>

  > Everything *is the same as* itself: $t = t$ is always true.
  >
  > Why? The idea is deceptivly simple.
  > Say there perhaps existed a thing t which was *not* the same as itself.
  > Then there's some property P which distinguishes t from itself.
  > In other words the statement Pt is true, but on the other hand Pt must also be false in order to distinguish t from itself.
  > This situation would certainly violate the Law of Law of Non-Contradiction, and so cannot occur.

Equality is "Symmetric"

> If $a = b$ then $b = a$.
>
> Why? Again, it's deceptivly simple.
> If a is the same thing as b, then by definition of *same*:
> All statements true for a are also true for b, and all statements statements true for b are also true for a.
> But this says the same thing as:
> All statements true for b are also true for a, and all statements statements true for a are also true for b.
> Which just means $b = a$, by definition of *same*.

Note: The much shorter version of this argument is to just observe that the definition of *identical* requires that "every statement which is true for one is also true for the other". So the order of the things a and b cannot make a difference in whether or not they are equal.

Equality is "Transitive"

> Given any three things a, b, and c, if $a = b$ and $b = c$, then $a = c$.
>
> Why?
> Exercise. It's pretty much the same as the previous two.

See:
`https://en.wikipedia.org/wiki/Identity_of_indiscernibles`
`https://en.wikipedia.org/wiki/Impeachment_of_Bill_Clinton`

- **Set, collection, group of things**
  A *set/collection* is a *thing/object* which is distinguished from non-set objects by having a notion of *membership*. More specifically:

  > A thing S is a *set/collection* if and only if it given any thing t it makes sense to ask whether or not t is a *member* (or *element*) of the set S.

  The relationship between a set and the things which are members of it is usually expressed as:

  > A collection *contains* its things.

  Or as:

  > The things are *elements/members of* the set.

  Less frequently, and unfortuantely:

  > The elements/members are *contained in* the set.

You should avoid this last usage of "contained in", as it is also sometimes used to express the relationship between a subset and an ambient set. Context matters!

Note again: A *set/collection* is also a *thing/object/entity*. Every collection is an object, but not vice versa.

- **Informal rules for working with our elementary notions of sets/collections**
  Important!

  > At this point, we are going to talk about a very informal kind of "baby" or "naive" set theory. I am explicitly *not* making an attempt to develop a rigorous and formalized set theory that you would study in a college course devoted to set theory.
  >
  > In particular, we won't need to worry about the difference between a *proper class* and a *set*.
  >
  > See: `https://en.wikipedia.org/wiki/Set_theory`

  All we will need to get going is a *very basic and intuitive* understanding, and here are the rules we need to get going:

  1. *Extensionality.*
     A set is *determined by* its elements:

     > Two sets have the *same members* if and only if they are the *same object*.

     In other words, if it's always the case that a thing is in one set exactly when it is in the other set, then not only are the two sets equal/the-same *as collections of their members*, but they are in fact *actually the same object*.

     The equals sign "=" is used in this way when it comes to sets. It says that two sets have the same members, and this is the same as saying they are the same objects.
  2. *Empty Set.*
     There is a set called *the empty set*, denoted by $\{\}$ or $\emptyset$, which has no members. Given anything t, it is *never true* that t is a member of the empty set.

     Exercise: Verify using *Extensionality* that there really can be only one empty set.

  3. *Singleton and Pairing Axioms.*
     If I have a *specific* thing a, there is a set whose members are exactly just that thing a.

     > "$\{a\}$" is a standard way of writing down this set.

For example, if $a = 3$, and S is the set $\{3\}$, then S has exactly one member, namely the number 3. The number 3 is in the set, and anything different from the number 3 is not in the set.

If I have two *specific* things a and b, possibly different but possible the same, I can talk about a set which has precisely those things a and b as its elements.

"$\{a, b\}$" is a standard way of writing down this set.

To clarify:
If $a = 3$ and $b = 42$, and S is the set $\{3, 42\}$, then S has two members, namely the numbers 3 and 42.
If $a = 42$ and $b = 42$, and S is the set $\{42, 42\}$, then S has only one member, the number 42.

$\{42, 42\}$ is the same set as $\{42\}$.

Exercise: Convince yourself using *Extensionality* that $\{a\} = \{a, a\}$.

4. *Subsets.*
If S and T are sets, then S is a *subset* of T when every member of S is also a member of T.
Sometimes people say that S *is contained in* T.
(Note: This usage of "*is contained*" is unfortunate, since it is also used to mean being an *element*).

If S is a *subset* of T, then T is sometimes referred to as the *ambient* set.

Examples:

 – $\{3\}$ is a *subset* of $\{3, 42\}$.
 – $\{3, 3\}$ is a *subset* of $\{3, 42\}$.
 – $\{3, 3\}$ is a *subset* of $\{3\}$.
 – It is *not the case* that$\{5\}$ is a *subset* of $\{3, 42\}$.

Important! A *subset* is not the same idea as an *element/member*. A subset is a collection of things which are *elements* of an ambient set. Also, the empty set $\{\}$ is a subset of every set.

5. *Subset Specification.*
If S is a set, and P is a *property*, then there is a *subset* T of S consisting precisely of those elements x of S where Px is *true*.

For example:

If S is the set of all positive or negative integers, and P is the property of being an even number, then T would be the subset
$T = \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$.

6. *Power Set.*
   If you have a set, then you can talk about a second set called the *power set* (associated with the original set) whose *elements* are precisely those things which are *subsets* of the original set.

   For example:

   > If S is the set $\{1, 2\}$ then *power set* T for S is the subset
   >
   > $$T = \{\{\}, \{1\}, \{2\}, \{1, 2\}\}$$
   >
   > .

7. *Unions and Intersections.*
   If you have some sets, you can talk about the *union* of those sets and the *intersection* of those sets.

   The *union rule* says:

   > If you're looking at some sets, then there exists a set called the *union* of these sets, which is a set whose elements are precisely those elements which *appear in at least one* of the sets you're looking at.
   >
   > Example:
   > Let's say I'm looking at these sets: $\{1, 2, 3\}$, $\{2, 3, 4\}$, and $\{3, 4, 5\}$.
   > The *union* of these is the set $\{1, 2, 3, 4, 5\}$.

   The *intersection rule* says:

   > If you're looking at some sets, then there exists a set called the *intersection* of these sets, which is a set whose elements are precisely those elements which *appear in every one* of the sets you're looking at.
   >
   > Example:
   > Let's say I'm looking at these sets: $\{1, 2, 3\}$, $\{2, 3, 4\}$, and $\{3, 4, 5\}$.
   > The *intersection* of these is just the set $\{3\}$.

   Important! Trying to visualize intersections and unions using Venn diagrams isn't that helpful if you have more than a handful of sets.

   A much better view on the situation is achieved by realizing that all that's really going is applications of *existentiality* and *universality*!

   Let's look at things from that perspective.

   Let's say that $\mathscr{S}$ (that symbol is a math script letter "S") is a collection, and every element of $\mathscr{S}$ is a *set*. In other words, $\mathscr{S}$ is a "set of sets". Also, suppose that $\mathscr{S}$ is *not empty*, so it has at least one element. (This last assumption is to avoid a counterintuitive situation regarding the *intersection*.)

The *union* of $\mathscr{S}$ is the set consisting of all things x which are in *at least one* of the sets s in $\mathscr{S}$. In other words, a thing x is in the union of $\mathscr{S}$ if and only if it is an element of at least one of the sets in $\mathscr{S}$.

Restated:
The *union* of $\mathscr{S}$ is the set consisting of all things x which have this property:

> Property of a thing x:
> x is an element of *at least one* member of $\mathscr{S}$.
>
> Restated property of a thing x:
> There exists an element of $\mathscr{S}$ of which x as a member.
>
> Restated property of a thing x:
> There exists an s which [is element of $\mathscr{S}$ and of which x is a member].

Upshot:

> The *union* of $\mathscr{S}$ is the set of all things x which have this property:
> There exists an s which [is element of $\mathscr{S}$ and of which x is a member].
> The symbol
> $$\bigcup_{s \in \mathscr{S}} s$$
> is used as shorthand for this.

Similarly for *intersections*:

> The *intersection* of $\mathscr{S}$ is the set of all things x which have this property:
> For *every* s, if s is an element of $\mathscr{S}$ then x is a member of s.
> The symbol
> $$\bigcap_{s \in \mathscr{S}} s$$
> is used as shorthand for this.

Usually, when working with a small numer of sets, say sets A, B, C, and D, then the union and intersection are instead typically written

$$A \cup B \cup C \cup D$$

and

$$A \cap B \cap C \cap D$$

- **"Hidden implications" in statements involving *universality***
  There are often *implications* lurking in statements expressing *universality*:

  > Dogs are *always* mammals

  Most mathematicians would agree that these four statements say the same thing:

1. Dogs are *always* mammals.
2. *Every* dog is a mammal.
3. For *every* thing x, if x is a dog, then x is a mammal.
4. For *every* w, [w being a dog] *implies* [w being a mammal].

In a little more generality, let's say that S is some kind of collection of things, and P is some *property.*

Mathematicians would also tend to agree that these say the same thing:

1. *Everything* in the collection S has property P.
2. For *every* w, if w is in the collection S, then w has property P.
3. For *every* w, if w is in S, then Pw is *true.*
4. For *every* w, [w in S] *implies* [Pw is *true*].

What's going on here is something closely related to the notion of *Universe of Discourse* or *Domain of Discourse.*

See:
`https://en.wikipedia.org/wiki/Universe_(mathematics)`
`https://en.wikipedia.org/wiki/Domain_of_discourse`
`https://en.wikipedia.org/wiki/Class_(set_theory)`

- **"Hidden conjunctions" in statements involving *existentiality***
There are often *conjuntions* (*"ANDs"*) lurking in statements expressing *existentiality*:

    At least one dog is white.

Most mathematicians would agree that these five statements say the same thing:

1. *At least one* dog is white.
2. *There exists* a dog which is white.
3. *There exists* a thing which is a dog *and* which is white.
4. *There exists* a thing x which is a dog *and* which is white.
5. *There exists* a thing w such that [w is a dog] *and* [w is white].

In a little more generality, let's say that S is some kind of collection of things, and P is some *property.*

Mathematicians would also tend to agree that these say the same thing:

1. *At least one thing* in the collection S has property P.
2. *There exists* a thing in the collection S which has property P.
3. *There exists* a thing which is in the collection S *and* which has property P.

4. *There exists* a thing w which is in the collection S *and* which has property P.
5. *There exists* a thing w such that [w is in S] *and* [Pw is *true*].

Again, this is very much related to the notion of *Universe of Discourse* or *Domain of Discourse*.

See:
`https://en.wikipedia.org/wiki/Universe_(mathematics)`
`https://en.wikipedia.org/wiki/Domain_of_discourse`
`https://en.wikipedia.org/wiki/Class_(set_theory)`

- **The natural numbers**
(The *natural numbers* are sometimes called the *"counting numbers"*.)

The quotes near the top from Leopold Kronecker and Stephen Cole Kleene pretty much say it all. It seems like it might be pretty tough to construct a formalized version of the natural numbers which does not *somehow* already rely on a basic and intuitive understanding of the naturals (or equivalent). And even so, what would've been accomplished?

Let's look at this a little more. See these for details about what's coming next:

`https://en.wikipedia.org/wiki/Peano_axioms`
`https://en.wikipedia.org/wiki/Set-theoretic_definition_of_natural_numbers`

Two standard approaches which are used to *define* the natural numbers "from scratch" are the *Peano axioms*, and the *von Neumann construction*.

Let's look first at the *Peano axioms*:

1. There is a thing we'll call "0" ("zero"). This is a "natural number".
2. For every natural number n, there is a "next one" which will be denoted by short-hand $S(n)$.
Think of S as a "successor function".
(To be rigorous: Even though we don't really have a good idea of what a "function" is yet, there is no real problem, since you could shoehorn in a definition of functions as certain sets of "ordered pairs".)
3. The number 0 is *not* the successor of anything.
More precisely, and in the jargon we've used already:
It is *not the case* that *there exists* something t where $0 = S(t)$.
4. If two natural numbers are different, then their successors are different.
5. There are two equivalent versions:
<u>Version A: Axiom of Induction</u>

If P is a property which is true for 0, and is true for $S(n)$ whenever it's true for n, then P is true for every natural number.

Version B: Extremal Clause

A totality of the natural numbers exists and is the *set* of things which can be obtained from 0 by "repeatedly applying" the successor function "finitely many times".

Axioms #1-4 are straightforward and cause no issue. But the situation is still not totally clear, because we need to somehow make sense of Axiom #5. Let's explore.

First, lets say you choose to use Version A: Axom of Induction for your fifth axiom.

> Notice that none of these five axioms assert the existence of a *set* of the natural numbers. Instead, they just show how you can build, *one-by-one* and starting with 0, larger and larger sets of numbers from 0 to some number. But to use the natural numbers as a basis for further math means that you have to be able to treat the totality of them as a *set*. There is no guarantee or assertion in the five axioms that such a *set* of natural numbers exists. The problem remains to find or build a *set* which actually satisfies the five axioms.

Second, lets say you instead want to use Version B: Extremal Clause for the fifth axiom.

> A standard definition of *finite set* is one which can be put in one-to-one correspondence with some beginning portion of the natural numbers from 1 to N, or one which can be "completely counted". It seems that the phrase "finitely many times", interpreting *finiteness* in this way, requires *already having* an underlying understanding of the natural numbers. To prevent circularity, you need to come at *finiteness* from another direction. But as it stands Version B of the fifth axiom is not satisfactory to completely define the natrual numbers.

Let's look some more at what we need to use Version A: Axom of Induction as the fifth axiom. We are left with the question of whether or not there even exists a *set* which satisfies the five axioms and so behaves like we want natural numbers to behave. We want a *model* that satisfies all five axioms. Can we find one?

The *von Neumann construction* provides such a model which satisfies the five *Peano axioms*... but you don't get it for free.

The *von Neumann construction* involves building a sequence of sets, each one a kind of "successor" to the one before, by performing clever unions of sets. The successor to the set a is the set $a \bigcup \{a\}$. The natural numbers are then defined to be the *smallest* set (read: "*intersection* of all such" sets) which contains the empty set and all its successor sets as members.

But now you have a new problem. You need some assurance that *there even exists one such set* which has the empty set and each successor set as members.

To guarantee the existence of such a set, you need to add in a new set theory axiom: the *Axiom Of Infinity.*

The *Axiom Of Infinity* asserts that there exists a set $\mathscr{N}$ which has the empty set as a *element*, and is "closed under" the successor operation: $a \bigcup \{a\}$ is an element of N whenever a is an element of $\mathscr{N}$.

See these:
`https://en.wikipedia.org/wiki/Mathematical_induction#Axiom_of_induction`
`https://en.wikipedia.org/wiki/Axiom_of_infinity`

The upshot is that if you want to try to use Peano or von Neumann to *define* something which behaves like the natural numbers you are already and innately familiar with, then here are your choices:

1. Start with Peano, using <u>Version B: Extremal Clause</u> for the fifth axiom.
   In this case you are basically trying to use a pre-existing and intuitive understanding of the naturals to *define* the naturals, unless you can somehow meaningfully redefine "finitely many times".
2. Start with Peano, using <u>Version A: Axom of Induction</u> for the fifth axiom.
   Then you don't have the guarantee that there even *exists* a set which satisfies the five axioms. You have to then come up with one. But doing so seems to require something like the von Neumann construction, where you wind up having to adopt the *Axiom Of Infinity.*

Pick your poison.

3 A Few Examples of Basic Reasoning

<u>Example #1</u>

ASSUMPTION 1: I know that Max is a dog.
ASSUMPTION 2: I know that being a dog *guarantees* being a mammal.
ASSUMPTION 3: I know that being a mammal *guarantees* being an animal.
CONCLUSION: I conclude that Max is a animal.

How can I say this?
It is easy and is totally straightforward.

<u>Reasoning for Example #1</u>

Max is a dog, and being a dog *guarantees* being a mammal.
I now know that Max is a mammal. (See the earlier item about *guarantees/implies*.)
Being a mammal *guarantees* being an animal.
So I conclude that Max is a animal.

3.1 Imagining Possible Scenarios

Example #2

ASSUMPTION 1: You know that Max is a dog.
ASSUMPTION 2: You know that being a car *guarantees* not being a dog.
CONCLUSION: You conclude that Max is not a car.

One way to get the conclusion would be to notice, by recalling what it means to *guarantee* something, that it is *impossible* to be both a car and a dog simultaneously, and then conclude that if something is a dog then it is *not* a car.

But let's take a different, and admittedly more convoluted approach.

You are allowed to use your *imagination* and work with *hypothetical* scenarios. When you are using your imagination to make and then "enter into" these hypothetical scenarios, you keep all of your original assumptions, but you tentatively add in new ones which you are testing out. The goal is to see whether or not the new assumptions "play nicely" with the oiginal assumptions. This approach is something that you will see done frequently when you are reading math.

Reasoning for Example #2

Main Argument:

1. I want to show that Max is not a car.
2. Is the scenario where Max *actually is* a car even possible, given the assumptions that we are making?
3. Let us imagine *"being in"* the scenario where Max is a car, and let's see how it plays out.
4. *"Enter into"* this imaginary scenario.

   Imaginary Scenario:

   5. We now have an additional assumption along with our original two assumptions.

      ASSUMPTION 3: Max is a car.

   6. From ASSUMPTION 2, being a car *guarantees* not being a dog.

7. From ASSUMPTION 2 combined with ASSUMPTION 3, we know that Max *must not be* a dog.
8. But from ASSUMPTION 1 we know that Max *actually is* a dog.
9. Max *both is and is not* a dog.
10. The scenario we are in violates *Non-Contradiction*.
11. *"Leave"* this imaginary scenario.

Main Argument:

12. The scenario we just imagined, where Max is a car, violates *Non-Contradiction*, can not actually occur, and so can be "ruled out".
13. By *Process of Elimination*, we can conclude that Max is not a car.

Here is another example similar to but a little more complicated than the one we just looked at.

Example #3

> ASSUMPTION 1: You know that Max is a dog.
> ASSUMPTION 2: You know that being a dog guarantees being a mammal.
> ASSUMPTION 3: You know that being a mammal guarantees being a creature.
> CONCLUSION: You conclude that Max is a creature.

The obvious (and easiest) way of getting our conclusion here is the mindless and straightforward approach:

> Max is a dog.
> Being a dog guaratees being a mammal.
> So Max is a mammal.
> Being a mammal guarantees being a creature.
> Therefore Max is a creature.

But instead, let's approach this one as we did the previous example, but with one extra "level" of imagination. This is definitely *not* the easiest or most straightforward approach. We are doing it this way just to see how you can *nest* one imaginary scenario inside another.

Reasoning for Example #3

Main Argument:

1. What would happen if Max where *not* a creature?
2. Now *imagine* a scenario where Max is not a creature, but everything else that we know is also still true.
3. Just for clarity let's call this imaginary scenario *Imaginary Scenario #1*.
4. *"Enter into" Imaginary Scenario #1*.

Imaginary Scenario #1:

5. We have all of our original knowledge/assumptions, plus the additional "fact" that Max is not a creature.
6. Max is not a creature.
7. Could Max possibly be a mammal inside this scenario?
8. Imagine yet another scenario, *Imaginary Scenario #2*, which "lives inside" *Imaginary Scenario #1*, where Max is a mammal.
9. *"Enter into" Imaginary Scenario #2.*

   Imaginary Scenario #2:

   10. We have all of our original knowledge/assumptions from *Imaginary Scenario #1*, plus the "fact" that Max is a mammal.
   11. Max is a mammal.
   12. But being a mammal guarantees being a creature.
   13. Max must therefore be a creature.
   14. Max both *is* and *is not* a creature, violating *Non-Contradiction*.
   15. This is impossible.
   16. *"Leave" Imaginary Scenario #2.*

Imaginary Scenario #1:

17. We conclude that *Imaginary Scenario #2*, where Max is a mammal, can not happen inside *Imaginary Scenario #1*.
18. By *Process of Elimination*, we conclude that Max *is not* a mammal inside *Imaginary Scenario #1*.
19. But we still know that being a dog guarantees being a mammal, and that Max is a dog.
20. We conclude that Max also *must be in fact* a mammal.
21. Max both *is* and *is not* a mammal, violating *Non-Contradiction*, inside *Imaginary Scenario #1*.
22. This is impossible.
23. *"Leave" Imaginary Scenario #1.*

Main Argument:

24. We have determined that *Imaginary Scenario #1*, where Max is *not* a creature, is impossible.
25. By *Process of Elimination*, we conclude therefore that Max *must in fact be* a creature.

The point of this example is that you can *nest* imaginary scenarios inside each other, as long as you successfully tie up loose ends, and don't lose track of where you are. You can go down rabbit holes, and navigate forks in paths, but you always have to go out the way that you went in.

And just to reiterate: Although *valid*, this way of approaching the third example is really not the best way. It is overly complicated and convoluted.

Using and nesting imaginary scenarios in order work out which scenarios are possible and which scenarios are impossible is very similar in flavor to these things:

- An ant exploring the branches of a tree trying to find food.
- Following forks in a path in the woods, making sure that you always go back the way you went in, and not jumping around.
- "Lexical scoping" in computer programming.
- "Stacks" in computer science.
- The confusing "nested tangents" that some people go on when they are telling long stories.

At this point you could skip to and understand the proof of *De Morgan Duality* in the later section *De Morgan Duality: Existentiality vs. Universality* further down.

Then come back here.

3.1.1 Don't get lost when you are exploring possibilities

Process of Elimination is a process involving three steps:

1. Identifying an exhaustive list of possible scenarios.
2. Exploring these possible scenarios.
3. Ruling out all but one of the possible scenarios.

Applying *Process of Elimination* is straightforward, almost second nature, and completely reasonable. I almost didn't even include the current section at all. But I did include it, just in case?

> "What could possibly go wrong"?
>
> -Anonymous

Well, what can go wrong is that you might "get lost" if you're not careful enough.

Here is a simple example to show one way this happens. It is intentionally confused in a way that shows up every so often when 1) you're wrestling with difficult math, or 2) you're debugging a confusing, complicated, and/or poorly-writen bit of computer code.

Let us try to prove that $1 = 2$.

1. Let $n = 1$.
2. Is the scenario possible wherein it is *not true* that $n = 2$?
3. In this scenario: It is *not true* that $n = 2$.
4. In this scenario: Let $m = 2$.
5. In this scenario: Either $n = m$ or it is *not* the case that $n = m$.
6. In this scenario: Suppose it *is true* that $n = m$.
7. In this scenario: Then, since $m = 2$ and $n = m$, we conclude that $n = 2$.
8. In this scenario: It both *is* and *is not* true that $n = 2$.
9. In this scenario: This violates *Non-Contradiction*.
10. The scenario wherein it is *not true* that $n = 2$ violates *Non-Contradiction*.
11. The scenario is *not* possible wherein it is *not true* that $n = 2$.
12. By *Process of Elimination*, we can conclude it must be the scenario where $n = 2$.
13. We conclude that $n = 2$.
14. But by assumption, we have that $n = 1$.
15. Both of $n = 2$ and $n = 1$ are *true*, and so we are forced to conclude that $1 = 2$.

Obviously, there is a huge problem here. The critical error occurs between Step 9 to Step 10. What exactly is the problem?

Here is the problem. We lost track of how many imaginary scemarios there were, and where we were in them. Starting at Step 6, there were not just *one* but actually *two* imaginary scenarios, one inside the other. Going from Step 9 to Step 10 we did not "go out the way that we came in". We incorrectly came out of two imaginary scenarios at the same time. This kind of sloppiness causes mistakes.

Here is the same argument, but formatted differently to clearly reveal the issue:

Main Argument:

1. Let $n = 1$.
2. Is the scenario possible wherein it is *not true* that $n = 2$?

    Let us imagine being in this scenario.

    Imaginary Scenario #1:

    3. In *Scenario #1*: It is *not true* that $n = 2$.
    4. In *Scenario #1*: Let $m = 2$.
    5. In *Scenario #1*: Either $n = m$ or it is *not* the case that $n = m$.

        Let us imagine being in the sub-scenario where $n = m$.

Imaginary Scenario #2: ("sub-scenario" of *Imaginary Scenario #1*)

6. In *Scenario #2*: Suppose it *is true* that $n = m$.
7. In *Scenario #2*: Then, since $m = 2$ and $n = m$, we conclude that $n = 2$.
8. In *Scenario #2*: It both *is* and *is not* true that $n = 2$.
9. In *Scenario #2*: This violates *Non-Contradiction*.

Main Argument:

10. The scenario wherein it is *not true* that $n = 2$ violates *Non-Contradiction*.
11. The scenario is *not* possible wherein it is *not true* that $n = 2$.
12. By *Process of Elimination*, we can conclude it must be the scenario where $n = 2$.
13. We conclude that $n = 2$.
14. But by assumption, we have that $n = 1$.
15. Both of $n = 2$ and $n = 1$ are *true*, and so we are forced to conclude that $1 = 2$.

The upshot, and the critical error, is this:

All we allowed to conclude from Step 9 is this:

> *Imaginary Scenario #2* is not possible *given the assumptions* provided by *Imaginary Scenario #1*.

In other words:

> If we are working inside *Imaginary Scenario #1*, then we can rule out *Imaginary Scenario #2* as a possible "sub-scenario" inside of *Imaginary Scenario #1*.

Step 9 *does not allow us* to just rule out *Imaginary Scenario #1* itself!

Incorrectly ruling out *Imaginary Scenario #1* itself was our mistake.

4 Some Important *Non-Elementary* Concepts

Here are several additional concepts, which appear very frequently. Each of these can be very conveniently formulated in terms of *elementary* items we've already seen.

- **unique, no more than one, at most one**
  - "There is *at most one* thing x with property P" means:
    If x and y both have property P, then x and y are *identical/equal/same* ($x = y$).
  - "There is *no more than one* thing x with property P" means the same thing.
  - "If there is an x with property P, then it is *unique*" means the same thing.
  - "If Px and Py then $x = y$" means the same thing.

The above four items *do not assert the existence* of a thing with property P, but rather only that if there *happens to be* something with property P, then any two which have that property are *indistinguishable*.

- "There is *exactly one* thing x with property P" means:
  There exists an x with property P, and if y also has property P, then it turns out to be *identical to* x.
- "*There exists one and only one* thing x with property P" means the same thing.
- "*There exists a unique* thing x with property P" means the same thing.
- "*There exists* an x with property P and it is *unique*" means the same thing.
- "*There exists* an x which satisfies Px is true, and for all y and z, if Py and Pz hold, then $y = z$" means the same thing.
- "*There exists* an x which satisfies (Px is true and for every y, if Py is true then $y = x$)" means the same thing.
  Note the compound propoerty thing going on there.

These six items assert both *existence* and *uniqueness*.

Upshot: *Uniqueness* does not by itself imply *existence*, but merely that *there can't be two or more* of something.

- **Proof by Contradiction**

> Reductio ad absurdum [Proof by Contradiction], which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.
>
> -G.H. Hardy, *A Mathematician's Apology*

We've actually already seen *Proof by Contradiction* earlier. The idea is simple:

> Let's say you believe that some statement is *true*, and you would like to actually prove it.
>
> From *Excluded Middle*, there are only two possibilities to consider:
>
> 1. the statement is either *true*
> 2. the statement is either *false*
>
> If you *imagine a scenario* in which the statement is hypothetically *false*, and then you discover as you work inside that scenario that *Non-Contradiction* is somehow violated inside that scenario, then you are forced to conclude that that scenario cannot really exist.
>
> In this way you determine that the scenario where the statement is *false* is impossible. By *Process of Elimination*, then, you are forced conclude that the statement must be *true*.

That's it. Here's an example.

> Let's say you and your friend agree that being a dog guarantees being a mammal.
> You and your friend agree that an iguana is not a mammal.
> But for some crazy reason your friend also believes that her iguana is actually also a dog.
> You need to convince your friend that her iguana is *not actually a dog*.

Here's a *Proof by Contradiction*:

1. Is it possible that the iguana is a dog?
2. *"Enter into"* the imaginary scenario where the iguana is a dog.
3. *Imaginary scenario:* The iguana is a dog.
4. *Imaginary scenario:* We still agree that being a dog guarantees being a mammal.
5. *Imaginary scenario:* We can conclude that the iguana *must be* a mammal, since it's a dog.
6. *Imaginary scenario:* But we still agree that an iguana *is not* a mammal.
7. *Imaginary scenario:* We now know that the iguana *both is and is not* a mammal.
8. *Imaginary scenario:* This violates *Non-Contradiction*.
9. *"Leave"* the imaginary scenario where the iguana is a dog.
10. The imaginary scenario violated *Non-Contradiction*.
11. We conclude that the scenario, where iguana is a dog, is *not actually possible*.
12. It is *not the scenario* that his iguana is a dog.
13. So his iguana is *not* a dog.

- **Proof by Cases**
  There's nothing fancy going on here. The reasoning can be justified using *Proof by Contradiction*.

  Proof by Cases:

  > Let A, B, and C be statements.
  >
  > Suppose you know :
  >
  > 1. If A is *true*, then C must be *true*.
  > 2. If B is *true*, then C must be *true*.
  > 3. At least one of A or B is *true*
  >
  > Then it *must be* the situation that C is *true*.

  In other words:

If there are two scenarios A and B (the so-called *cases*) where at least one of them must occur, and some third scenario C which is guaranteed by A and also by B, then scenario C is guaranteed.

Here's an example using *Proof by Cases.*

You know:

1. Due to COVID-19 restrictions, the local grocery store is open only on Monday and Thursday this week, and you are allowed to go on only one of those days.
2. The store gets shipments of hand sanitizer on Monday, which are sold out by the end of the day.
3. They won't have any fruit until Thursday.

You conclude that this week you can't get both fruit and hand sanitizer.

Why?
If you go on Monday ("CASE 1"), you can get hand sanitizer but not fruit.
If you go on Thursday ("CASE 2"), you can get fruit but not hand sanitzer.
These two cases "exhaust the possibilities" for when you can go to the store.
In either case, you wind up coming home with only one item.

Here's another example of *Proof by Cases.*

Let n be any positive or negative integer.
Let's prove that $||n| - 10| \leq |n - 10|$.

There are three cases to look at, which are provided by $n \geq 0$, $-10 < n < 0$, and $n \leq -10$.

CASE 1: Suppose $n \geq 0$.

Then $|n| = n$, so $||n| - 10| = |n - 10|$.
The conclusion that $||n| - 10| \leq |n - 10|$ is immediate.
Done with this case.

CASE 2: Suppose $-10 < n < 0$.

Then $|n| < 10$. So

$$||n| - 10| = |10 - |n|| = 10 - |n| < 10 + |n| = |10 + |n|| = |10 - n|$$

Done with this case.

CASE 3: Suppose $n \leq -10$.

Then $|n| \geq 10$. So

$$||n| - 10| = |n| - 10 < |n| + 10 = ||n| + 10| = |-n + 10| = |n - 10|$$

Done with this case.

No matter what value of n you are looking at, it falls into one of these three cases, and the desired inquality is satisfied in all cases.

Let us try to justify *Proof by Cases* by using *Proof by Contradiction.*

Let A, B, and C be as above.

1. I want to show that C is *true.*
2. *Imagine/assume* to the contrary that C is *false.*
3. Consider statement A.
4. We know that the truth of A *implies* the truth of C.
5. Recalling what *"implies"* means, it is impossible to have A *true* and C *false.*
6. We conclude that A must be *false.*
7. Now consider statement B.
8. In the same way we can conclude that B is also *false.*
9. We have shown that *both* of A and B are *false.*
10. But by our original assumptions, *at least one* of A or B is *true.*
11. Lines 9 and 10, taken together, violate *Non-Contradiction.* (Recall *AND/OR Duality*, and also see note below.)
12. The assumption that C is *false* results in a contradiction, and so is incorrect.
13. C must be *true.*

Important! Recall the idea of *AND/OR Duality*:

At least one of two statments is true if and only if they're *not both* false.

Make sure to understand why lines 9 and 10, taken together, violate *Non-Contradiction.*

- **implication/convese/inverse/contrapositive**
Let A and B be statements, and consider the third statement

A implies B

This kind of statement is called an *implication.*
The *contrapositive* of the original implication is the statement

(not B) *implies* (not A)

The *converse* of the original implication is the implication

B implies A

The *inverse* of the original implication is the implication

(not A) *implies* (not B)

(Note that the *inverse* is the *contrapositive* of the *converse*.)

- **Equivalence of an implication with its contrapositive**
  Burn this into your brain (metaphorically, of course):

  > An implication is *true* if and only if its *contrapositive* is *true*.

  Here's why:

  > Recall that
  >
  > *A implies B*
  >
  > is *false* exactly in the scenario where A is *true* and B is *false*.
  > This is the exactly the scenario where (not A) is *false* and (not B) is *true*.
  > But this is the same scenario where the *contrapositive*
  >
  > (not B) *implies* (not A)
  >
  > is *false*.
  >
  > *Note: Excluded Middle is lurking here in the equivalence between an implication and its contrapositive... Can you see how?*

  Make sure you become completely fluent going back and forth instantly between an implication and its contrapositive! It's done all over the place in math without even being mentioned.

  In a nutshell, these two say/mean the same thing:

  1. being a dog guarantees being a mammal
  2. not being a mammal guarantees not being a dog

- **ordered pairs**
  An *ordered pair* is something that "packages up" two different or identical things into one thing, in such a way that the order of the things is taken into consideration.

  If x and y are two (not necessarily different) things, then the *ordered pair* "(x, y)" is defined to be:
  $$(x, y) = \{\{x\}, \{x, y\}\}$$
  You can check, using *Extensionality*, that $(x1, y1) = (x2, y2)$ exactly if $x1 = x2$ and $y1 = y2$.

  This last property is what captures the idea that the two objects are in some sense "ordered". It's known as the so-called *characterizing property* of ordered pairs, and it's really the only thing you need to know about them in order to work with them.

  The actual definition used for ordered pairs really doesn't matter very much at all.

It's just a kind of "programming". In fact, many people like to think of ordered pairs as *elementary*, and actually this view really resonates with me. Even though we can *define* ordered pairs as special sets (like we did above), all the various ways of actually doing so apparently just seek to capture a notion of "order" that we already intuitively have:

> I have a first thing x in my left hand, and a second thing y in my right hand.

If (x, y) is an ordered pair, then x is often called the *first entry/component/coordinate* and y is often called the *second entry/component/coordinate*.

See: `https://en.wikipedia.org/wiki/Ordered_pair`

- **relations**
  A *relation* is just a *set* of *ordered pairs*.

  This is an example of a *relation*:

  $$\{(Max, dog), (Max, cat), (Fido, dog), (Bill, dog), (Lucy, cat), (Smallz, hamster)\}$$

  Notice that the name *Max* appears in the first positions of two pairs, and the animals *dog* and *cat* appear in the second positions of those same pairs. *Max* is "associated with" *dog* and *cat*.

  Similarly, the animal *cat* appears in the second position two pairs, and is associated with the names *Lucy* and *Max*.

  Here's another example.

  > Let a *relation* S be the set consisting of all ordered pairs (p, b) where p is a person, and b is a place where that person has at some point been employed.
  >
  > So an ordered pair (p, b) is an *element* of the relation S if and only if p is a person who has worked at the business b.
  >
  > Some people may have worked at multiple businesses, so there might be more than one ordered pair whose first entry is that person.
  >
  > Also, some businesses might have had more than one employee, so there might be more than one ordered pair whose second entry is that business.

  Here's a third example.

  > Let's say you draw square on graph paper.
  >
  > The set consisting of all the points (x, y) which are inside that square is a *relation*.

- **domain/range**
  The *domain* of a relation is the set of all things which are *first entries* of ordered pairs in the relation. The *range* of a relation is the set of all things which are *second entries* of ordered pairs in the relation.

  Example:

  > Let R be the relation which is the following set of orderd pairs:
  >
  > $$R = \{(1, 10), (2, 11), (3, 12), (4, 13), (5, 14)\}$$
  >
  > The *domain* of R consists of the numbers 1 through 5.
  > The *range* of R consists of the numbers 10 through 14.

- **functions**
  A *function* is just a *relation* which satisfies one additional requirement:

  > A relation is a *function* if and only if whenever (x, y) and (x, z) are both in the relation, then $y = z$.

  Here are three ways to rephrase the requirement:

  1. If two ordered pairs in the relation have the same first entry, then they also have the same second entry.
  2. For every thing x in the relation's *domain*, there is exactly one thing y in the domain's *range* which is associated with x.
  3. For every thing x in the relation's *domain*, *there exists a unique* thing y such that (x, y) is in the relation.

  Many people were taught to think of functions as "machines" which take an input and turn it into an output, or to think of them as "formulas" such as $f(x) = x^2 - x - 1$, as opposed to a collection of ordered pairs. While this kind of mental model might be good enough for high-school-level math or maybe engineering, it is the wrong one for studying math.

  But if you think about it for a moment, you'll realize that – since a function is a *set of ordered pairs* – it is actually the exact same thing as something you've likely seen and worked with before: "the *graph* of a function".

  In mathematics, a function is the *same thing* as its graph. In programming jargon, functions are "first-class citizens". They are *not* merely machines, formulas, or symbols that just give you a way to *draw* a graph.

  (One example of where it's quite inconvenient to think of functions as formulas or machines is when you are trying to piece together a fuction on a larger domain from functions on smaller domains which agree on overlapping patches... as you do all the time in analysis and topology.)

Important!

> You will often see someting like:
>
> $$\text{Let } f : X \to Y \text{ be a function.}$$
>
> You'll see this, for example, in Walter Rudin's *Principles of Mathematical Analysis*.
>
> This says these three things:
>
> 1. f is a *function*, and so by definition is also a *relation*.
> 2. The *domain* of the relation f is the set X.
> 3. The *range* of the relation f is *a subset of but not necessarily all of* Y.
>
> In this context, Y is typically called the *codomain*, to distinguish it from the *range*. The important thing here is that the *codomain* can be any set which is *large enough* to contain the *range* of the relation f.
>
> Some mathematicians think of the codomain as part of the *definition* of a function, and that changing the codomain yields a different function.
>
> Other mathematicians think of a function as just a set of ordered pairs with one additional requirement (like I defined it above), and that changing the codomain doesn't change the function *per se*, but rather changes only how you should *interpret* items in the function's range.
>
> It is matter of taste, and you should know which convention an author is using. It's possible to get caught off guard here.

- **Some notation used in the context of relations and functions**
  Here is some standard notation you will see when people are writing about realtions and functions.

> If Q is a relation,
> then the notation "aQb" *is often* used to mean that $(a, b) \in Q$.

Functions are relations, and so the same notation can also be used for functions:

> If G is a function,
> then the notation "sGt" *is sometimes* used to mean that $(s, t) \in G$.

But more commonly, in the case of functions:

> If G is a function, and s is in the *domain* of G,
> then the notation "$G(s)$" *is very commonly* used to indicate the *unique* thing t such that $(s, t) \in G$.

If G is a function, then given any s in its domain, there is one and only one thing which is associated with s via G. "$G(s)$" (or sometimes "Gs" without the parentheses) is the *notation* employed to name this thing.

In other words:

$(s, G(s))$ is the only ordered pair with first component equal to s in the realtion G:

$$(s, G(s)) \in G$$

Yes, this looks weird, because the symbol "G" is used in two different ways:

1. First, to represent a relation (in this case one which is also a function).
2. Second, as part of the more complicated symbol "G(s)" to indicate a thing associated with s via the function G.

- **Warning: Symbols are not functions!**
  Symbols and expressions (complicated symbols made up of other symbols) *are not* the same thing as functions!
  The upshot is this:

  - The *symbol* "$x^{1/n}$" is *notation* which is often used to specify the n-th root of whatever is specified by the symbol "x".
    The *symbol* "$x^{1/n}$" is *not* a function.
  - A *function* is something which provides a way of associating or relating elements of two sets.
  - But the symbol "$x^{1/n}$" can be used to help specify a *function.* In fact, the function which *maps* positive numbers to their n-th roots can be specified, for example, like this:
    $$\{(x, x^{1/n}) : x \text{ is a positive number}\}$$
    or like this:
    $$\text{The map } x \mapsto x^{1/n} \text{ where } x \text{ is a positive number}$$
  - Similarly, the *symbol* "$42t^2 + 7t + 1$" is *not* a function, but it can be used to help specify a *function*, for example like this:
    $$\{(t, 42t^2 + 7t + 1) : t \text{ is a complex number}\}$$
    or like this:
    $$\text{The map } t \mapsto 42t^2 + 7t + 1 \text{ where } t \text{ is a complex number}$$

  You might be tempted to think that the next example provides a situation where a symbol *actually is* a function:

  Let $G : \mathbb{C} \to \mathbb{C}$ be given by $G(z) = z^2$.

Now the symbol "G" is a function, right?

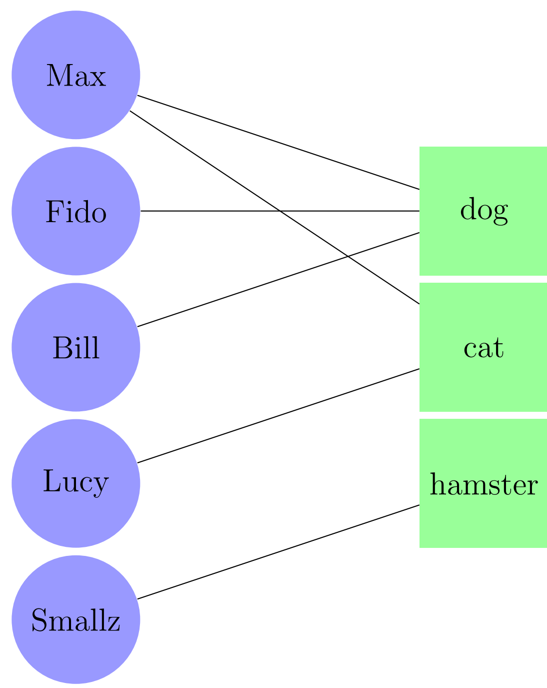
Nope.
The symbol "G" *refers to/is a name for/is a symbol for* the function $z \to z^2$, but is *not itself* a function.

- **Relations as graphs**

The relation

$$\{(Max, dog), (Max, cat), (Fido, dog), (Bill, dog), (Lucy, cat), (Smallz, hamster)\}$$

can be expressed in graph form like this:



The elements in the first positions of the ordered pairs are lined up vertically without repeats on the left, and the elements in the second positions of the ordered pairs are lined up vertically without repeats on the right.

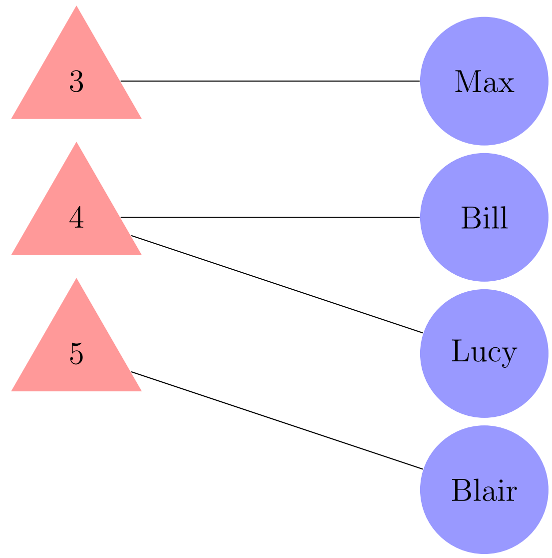
Grpahs consist of *nodes* and *edges*. In the above graph, the nodes are the names, and the types of animals. The edges are the lines connecting the nodes.

- **Composition of relations**

Suppose we now also specify relation between lengths of names and the names themselves as follows:

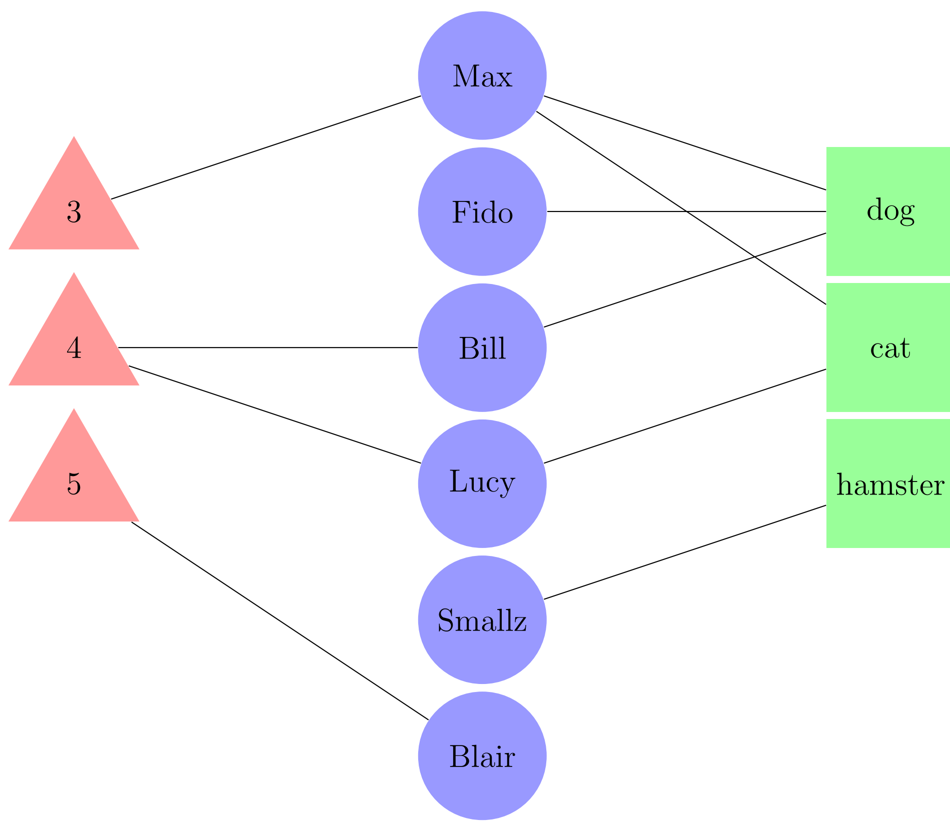
$$\{(3, Max), (4, Bill), (4, Lucy), (5, Blair)\}$$

This can be expressed in graph form as



We can combine the name/animal graph and the name/length graph in to a three-column graph, by including all of the *nodes* (names, animal species, and name lengths) which appear in either graph, and then including all the *edges* (the lines connecting the nodes) which appear in either graph.

Doing so results in this graph:

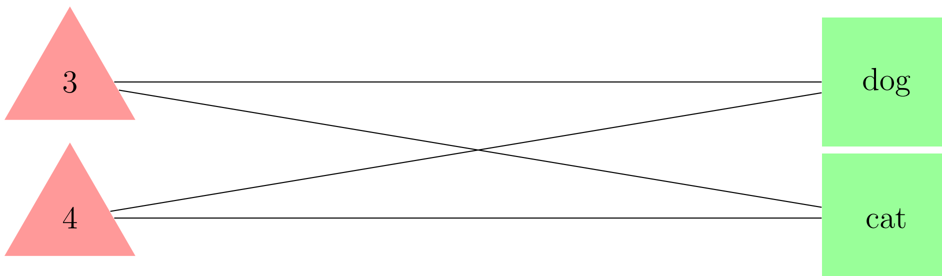


Notice that you can follow an edge from the node “3” to the node “Max”, and then

from the node “Max” to the node “dog”. This is because
$(3, Max) \in \{(3, Max), (4, Bill), (4, Lucy), (5, Blair)\}$
and
$(Max, dog) \in \{(Max, dog), (Max, cat), (Fido, dog), (Bill, dog), (Lucy, cat), (Smallz, hamster)\}$.

This process of following edges from one side to the other suggests a way to form a new relation, called the *composite* of the two relations.

Graphically, the ordered pairs of the *composite* relation are the pink triangles and the green squares which can be can be reached from each other by travelling along edges. Here is the graph of this relation:



Notice that the triangle “5” and the square “hamster” do not appear.
“5“ connects only to “Blair”, which does not have an animal associated with it.
“hamster” connects only to “Smallz”, which does not have a length associated with it.

The official definition of the *composite* of two relations is as follows:

> If R and S are relations, then we may form another relation, called “the *composite* of S and R”, or “R followed by S”, and written “$S \circ R$”, consisting of all ordered pairs (x, y) where *there exists* some t such that $(x, t) \in R$ and $(t, y) \in S$.
>
> *Exercise: Apply this definition to the relations in the previous graphs and verify that it successfully captures the idea of travelling along edges from one side to the other.*

- **Composition of functions**
Since functions are just special types of relations, the composite of two functions is defined the same way as for relations in general.

Recall the criterion for a relation to be a function:

> A relation is a *function* if and only if whenever (x, y) and (x, z) are both in the relation, then $y = z$.

Fact: The Composite Of Two Functions Is A Function

Proof:
Let F and G be functions, and let both (x, y) and (x, z) be in $G \circ F$. Then there exists some s and there exists some t such that

$$(x, s) \in F \text{ and } (s, y) \in G$$

and

$$(x, t) \in F \text{ and } (t, z) \in G$$

But since F is a function this means that $s = t$, so we know

$$(t, y) \in G \text{ and } (t, z) \in G$$

We also know that G is a function, so $y = z$.
This shows that $G \circ F$ is a function.

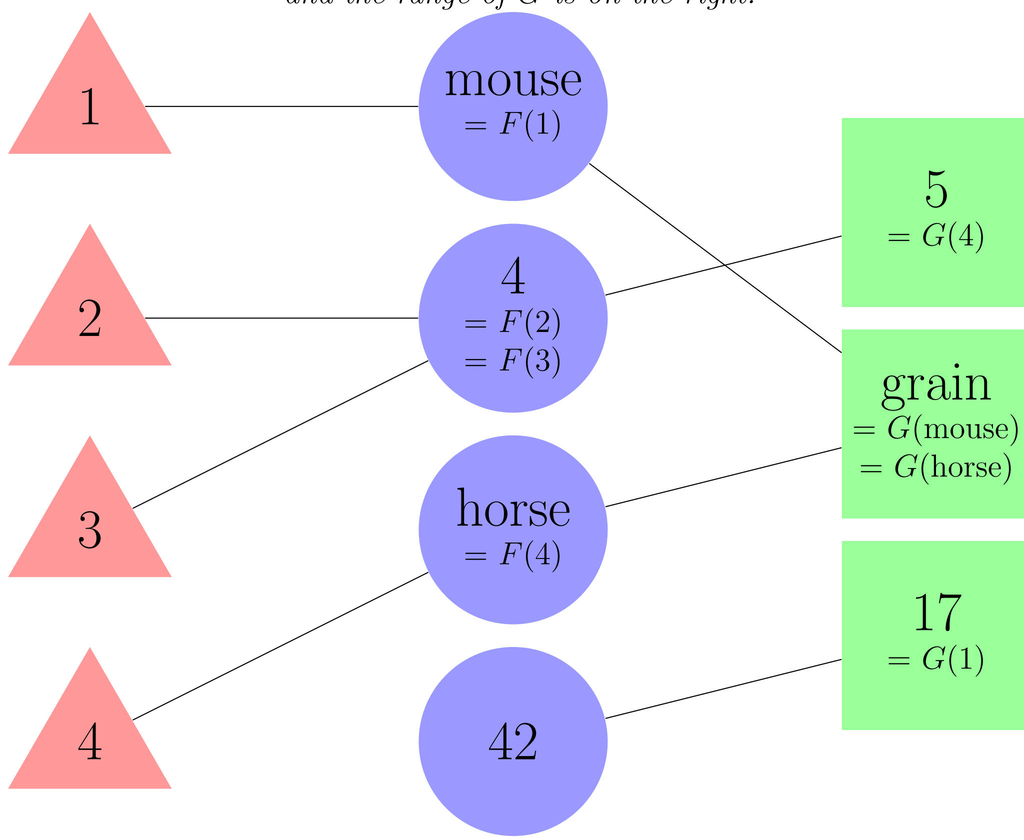
Consider the following two functions:

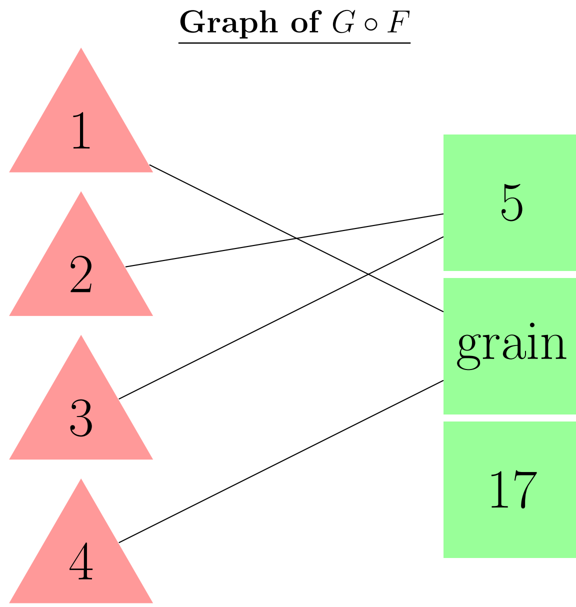
$$F = \{(1, \text{mouse}), (2, 4), (3, 4), (4, \text{horse})\}$$
$$G = \{(\text{mouse}, \text{grain}), (4, 5), (\text{horse}, \text{grain}), (42, 17)\}$$

Let's combine the graphs of these functions into one graph, as we did previously.

Combined graph of F and G

The domain of F is on the left, the range of F and domain of G are in the middle, and the range of G is on the right.





The composite function $G \circ F$ is the set

$$G \circ F = \{(1, \text{grain}), (2, 5), (3, 5), (4, \text{grain})\}$$

For example, the ordered pair $(2, 5)$ is in $G \circ F$ because there exists a t (namely $t = 4$) such that $(2, t) \in F$ and $(t, 5) \in G$. This relationship appears in the above graph **"Combined graph of F and G"** as a path first from the red triangle "2" to the blue circle "4", then from the blue circle "4" to the green square "5".

- **More about composition of relations and functions**
  Please see the appendix section "*More on Composition of Relations and Functions*"

5 Some Jargon

- **"holds"**
  This just means "is *true*".

  Given a statement or property S,

  "*S holds*"

  means the same thing as

  "S is *true*".

- **if A is true then B is true**
  Means the same thing as:

  1. A *implies* B.

2. Impossible to have A and not B.
3. Either A is false, or B is true.
4. If not B, then not A (the *contrapositive*)
5. B *follows from* A.
6. B *is necessary for* A.
7. B is true when A is true.
8. B is true if A is true.
9. B holds *as soon as* A holds.
10. A *requires* B.
11. B *is required for* A.
12. A *guarantees* B.
13. A is true *only if/when* B is true.

- **A is true if and only if B is true**
  Means the same thing as:

  1. A is true *"i.f.f."* B is true. (I know this is lame. But it is a standard shorthand.)
  2. Both "A implies B" and "B implies A" hold.
  3. A and B are *simultaneously true.*
  4. A holds *exactly/precisely if/when* B holds.

- **A is true except if B is true**
  Means the same thing as:

  1. (not B) implies A.
  2. Impossible to have (not B) and (not A).
  3. If (not B) is true then A is true.
  4. If B is false, then A is true.
  5. If A is false, then B is true (the *contrapositive*)
  6. A is true *unless* B is true
  7. A is true *except when* B is true.

  Note: Please see the section about the meaning of the words "unless" and "except"!

- **"neither" A "nor" B are true**
  Means the same thing as:

  1. A is false, and also B is false.
  2. *not* (A *or* B)

- **“vacuously” true**
  This is an idea and phrase that appears all over the place, and can be confusing until you get used to precisely what it means.

  Recall from the item about *implication* that “*A implies B*” is *true* in these three (of the four possible) *true/false* combinations:

  1. A is *true* and B is *true*
  2. A is *false* and B is *true*
  3. A is *false* and B is *false*

  An *implication* “*A guarantees B*” is said to be *vacuosly true* when A is *false*.

  So this *implication* is a *vacuosly true* statement:

  > [My name is Bryan and also my name is *not* Bryan] *guarantees* [Earth has 42 moons].

  “*Vacuosly true*” applies only to statements which involve (or can be thought of in terms of) *implications*.

  The next statement, which involves both *universality* and also *implication*, would also be said to be *vacuosly true*:

  > For *every* thing x, if x is a flying dog from another planet, then x is a Jedi.

  In general, let P and Q be properties, and suppose that for some reason, it turns out that for *every* x, Px winds up being *false*.

  Then the following statment would be considered to be *vacuously true*:

  > For *all* z, Pz *implies* Qz.
  >
  > Exercise: Use the technique for demonstrating the *truth* of a *universal* statement to prove that this *universal* statement actually is in fact *true*.

  Finally, let S be some set, and Q some property, and suppose it turns out that S is the *empty set*.

  Then this statment would be considered to be *vacuously true*:

  > *Everything* in S has property Q.

  Why? Recall that there is a *hidden implication* going on here, because

  > *Everything* in S has property Q

  says the same thing as

For *every* w, if w is an *element* of S, then w has property Q

But since the set S is *empty*, the statement

w is an *element* of S

is *false* for *every* thing w.
The implication

(w is an *element* of S) *guarantees* (w has property Q)

is *true* for every w.

See: `https://en.wikipedia.org/wiki/Vacuous_truth`

- **X is "of the form"** $\langle\langle\langle$***pattern***$\rangle\rangle\rangle$
A thing X is "of the form" $\langle\langle\langle$*pattern*$\rangle\rangle\rangle$ i.f.f. *there exists* an example which somehow matches the $\langle\langle\langle$*pattern*$\rangle\rangle\rangle$ and also equals X.

  Example: 12 is of the form *even* · *odd*, because $12 = 4 \cdot 3$.

- **Y "depends only on" X, Y "constant with respect to" Z**
Equivalent jargon:

  - Y is "determined by" X.
  - Y can be "recovered from" X.
  - Y is "a function of" X.
  -

  Let's suppose:

  1. You have three sets Z, X, and Y.
  2. You have a *relation* R_X whose *domain* is Z and whose *range* is X.
  3. You have a *relation* R_Y whose *domain* is Z and whose *range* is Y.
  4. You are guaranteed that:
  $y_1 = y_2$ whenever $(z_1, x) \in R_X$, $(z_2, x) \in R_X$, $(z_1, y_1) \in R_Y$, and $(z_2, y_2) \in R_Y$.

  In this case, it's often said that "Y depends only on X", "R_Y depends only on R_X", or "Y is constant with respect to Z".

  The crucial thing going on here is that the following *composite relation* is actually a *function* with domain X and range Y:

  The set of all ordered pairs (x, y) such that
  there exists some z with $(z, x) \in R_X$ and $(z, y) \in R_Y$.

If f and g are functions on common domain Z, and $g(z_1) = g(z_2)$ guarantees $f(z_1) = f(z_2)$, then you can say "f depends only on g".

Exercises:

1. Verify that if R_Y depends only on R_X then the relation R_Y must actually in fact be a *function*.
2. Verify that in the above scenario with f and g, there actually exists a function h defined on the range of g, such that $f = h \circ g$.

- **Y "doesn't depend on" Z, "constant with respect to" Z**
  This can mean two different but related things, depending on context.

  (i) More commonly:

  In the context where R is a relation (especially a *function*) with domain Z and range Y, "Y doesn't depend on Z" usually means that the range Y of the R consists one and only one object.

  In other words, all objects in Z are associated via the relation with the exact same thing.

  Said again, Y consists of precisely one thing.

  Yet another way of saying this is as follows:

  For every y_1, y_2, z_1, z_2,
  $y_1 = y_2$ whenever $(y_1, z_1) \in R$ and $(y_2, z_2) \in R$.

  (ii) Less commonly:

  It can mean that Y is "constant with respect to" Z, in the sense of the previous item (see **Y "depends only on" X**).

  You might occasionally see this usage in the context where you have relations R_X and R_Y, whose domains are both Z and whose ranges are respectively X and Y.

  But you're more likely to see this usage when the two relations R_Y and R_X were obtained from, or have been combined into, a so-called *ternary relation*.

  A ternary relation is just a set of "*ordered triples*"), and the ternary relation in question is:

  The set of all ordered triples (z, y, x)
  such that $(z, y) \in R_Y$ and $(z, x) \in R_X$.

- **Z "can be solved for" X "in terms of" Y, X is "defined implicitly" by Z and Y**
  Roughly:

  > For a given/fixed value Z, and every value Y, there exists a *unique* X which "works".

  For example, saying that $P(x, y) = z$ "can be solved for x in terms of y" means that whenever you are given specific values for y and z, then there is a *unique* x such that $P(x, y) = z$.

  In this case, people also say that $P(x, y) = z$ "implicitly defines" x as a function of y and z.

Appendices

A Supplemental Stuff

A.1 Relationship between *Implication Elimination* and *Process of Elimination*

I like to think of both of these as *elementary*, but actually either one can be justified using the other. There is nothing fancy going on here. *Implication Elimination* and *Process of Elimination* are tied together by the equivalence of "A *implies* B" and "(not A) *or* B".

Justifying *Implication Elimination*, assuming that *Process of Elimination* is valid:

1. Let A and B be statements.
2. Assumption 1: The statement "A *implies* B" is *true*.
3. Assumption 2: A is *true*.
4. We seek to show that B must be *true*.
5. From Assumption 1, we know that either A is *false* or B is *true* or both.
6. Step 3 provides us with two scenarios, at least one of which *must* occur:
   (a) A is *false*
   (b) B is *true*
7. From Assumption 2 and *Non-Contradiction*, we know that the scenario (a) can *not* occur here.
8. Using *Process of Elimination*, we conclude that scenario (b), where B is *true*, *must in fact* occur.
9. We have concluded that B is in fact *true*.
10. QED

Justifying *Process of Elimination*, assuming that *Implication Elimination* is valid:

1. Let A and B be statements.
2. Assumption 1: There are two scenarios, at least one of which *must* occur:
   (a) A is *true*
   (b) B is *true*
3. Assumption 2: Scenario (a), where A is *true*, does not occur.
4. We seek to show that scenario (b), where B *true*, must occur.
5. Let C be the statement "A is *false*".
6. By *cancelling double negatives*, C is *false* exactly when A is *true*.
7. By Assumption 1, then, either C is *false*, or B is *true*, or both.
8. Step 7 just says that C *implies* B.
9. From Assumption 2, we know that statement C is *true*.
10. By applying *Implication Elimination* to Steps 8 and 9 together, we can conclude that B must be *true*.
11. We have determined that B is *true*.
12. QED

A.2 De Morgan Duality: Existentiality vs. Universality

Theorem. (De Morgan Duality)
Let P be a *property*. Exactly one of **A)** or **B)** holds:

- **A)** Px is *true* for *at least one* thing x. (Something has property P.)
- **B)** Px is *false* for *every* thing x. (Everything *fails* to have property P.)

Proof. First, let's show that it is *impossible for both* of **A)** and **B)** to hold. To do this we will use *Proof by Contradiction*.

1. Assume (to obtain a contradiction later) that both **A)** and **B)** hold.
2. Then by **A)**, Px is *sometimes* true.
3. Let a be such an example of a thing where Pa is *true*.
   (Recall how *applying/using existentiality* works.)
4. But by **B)** (Px is always false), we may *apply universality* to conclude that Pa is *false*.
5. We see that Pa is both *true* and textitfalse, and this violates *Non-Contradiction*.
6. This contradiction shows that our assumption (namely that both **A)** and **B)** hold) must be incorrect.

7. Thus *at most one* of **A)** and **B)** holds.

We just showed that *at most one* of **A)** or **B)** holds.

Next, we must show that *at least one* of **A)** or **B)** holds. Let's use *Proof by Cases* for this.

CASE 1: Statement **A)** holds.
CASE 2: Statement **A)** does not hold.

By *Excluded Middle*, at least one of *CASE 1* or *CASE 2* must occur.

If *CASE 1* occurs:

Then statement **A)** holds, so *at least one* of **A)** or **B)** holds.

The desired conclusion holds in *CASE 1*.

If *CASE 2* occurs:

Then statement **A)** *does not* hold. Let us show that then statement **B)** must hold.

Let b refer to some *arbitrary* thing. Let's show that Pb must be *false*, by using a quick *Proof by Contradiction*.

Assume for a contradiction later that Pb is *true*.

The presence of the thing b where Pb is *true* means we've met the requirement for *demonstrating existentiality* for the statement

Px is *true* for *at least one* thing x.

and thus statement **A)** holds.

This is a contradiction.

This contradiction allows us to conclude that Pb is *false*. But b was arbitrary! We have therefore met the requirements of *demonstrating universality* for the statement

Px is *false* for *every* thing x.

But this just means that statement **B)** in fact holds, so *at least one* of **A)** or **B)** holds.

The desired conclusion also holds in *CASE 2*.

We have successfully shown that the desired conclusion holds both in *CASE 1* and also in *CASE 2*.

This completes the proof.

□

A.3 De Morgan Duality, Again: Unions vs. Intersections

You might want to briefly revisit the section about *sets* and then return here.

Let $\mathscr{S}$ be a non-empty *collection* of *sets*. In other words, $\mathscr{S}$ is a *set*, $\mathscr{S}$ has at least one element in it, and every element of $\mathscr{S}$ is also itself a set.

Recall the definition of the *union* of $\mathscr{S}$:

> The *union* of $\mathscr{S}$ is the set of all things x which have this property:
>
> > there exists an s such that $s \in \mathscr{S}$ and $x \in s$.
>
> The symbol
> $$\bigcup_{s \in \mathscr{S}} s$$
> is used as shorthand for the *union* of $\mathscr{S}$.

Recall also the definition of the *intersection* of $\mathscr{S}$:

> The *intersection* of $\mathscr{S}$ is the set of all things x which have this property:
>
> > for *every* s, if $s \in \mathscr{S}$ then $x \in s$.
>
> The symbol
> $$\bigcap_{s \in \mathscr{S}} s$$
> is used as shorthand for the *intersection* of $\mathscr{S}$.

Suppose there is another set U, and every set in $\mathscr{S}$ is a *subset* of U. Think of U as providing a kind of ambient space.

Now, if T is any subset of U (for example T might be one of the members of $\mathscr{S}$), then you can form another set, shorthand $\neg T$, whose elements are precisely those elements of U which are *not elements of* T. This action is justified by the *Subset Specification* rule. $\neg T$ is typically called the *complement* of T in U. It's what you get when you "remove" elements of T from U.

Prove this as an exercise:

> The complement of the complement is the original set. $T = \neg(\neg T)$.

Given a set s which is an element of $\mathscr{S}$, we can form its complement $\neg s$. In fact, we can create a whole new *collection* of *sets* $\mathscr{C}$ whose elements are precisely all the complements $\neg s$ for things s in $\mathscr{S}$. In other words, something is an element of $\mathscr{C}$ if and only if it is $\neg s$ for some s in $\mathscr{S}$.

(Note: If you want, you can take it on faith that we are allowed to do this. But otherwise, go ahead and justify this action to yourself using *Subset Specification* applied to the *power set* of U. The property P involved is:

A thing x has property P if there exists an $s \in \mathscr{S}$ such that $x = \neg s$.)

Let us now consider how the following two sets are related:

1. $\neg \bigcup_{s \in \mathscr{S}} s$, the complement of the union of $\mathscr{S}$, and
2. $\bigcap_{s \in \mathscr{S}} \neg s = \bigcap_{t \in \mathscr{C}} t$, the intersection of the complements of things in $\mathscr{S}$.

By definition of intersection, a thing x is an element of $\bigcap_{t \in \mathscr{C}} t$ if and only if x has this property:

For *every* t, if t is an element of $\mathscr{C}$ then x is a member of t.

We can see (*as you should verify* by both applying and demonstrating *universality*) that the above property is *true* if and only if the following rephrasing is *true*:

For *every* s, if s is an element of $\mathscr{S}$ then x is a member of $\neg s$.

Let P be this property; i.e. Px is *true* for a thing x exactly when:

For *every* s, s being an element of $\mathscr{S}$ *guarantees* x is a member of $\neg s$.

We have shown:

The set

$$\bigcap_{t \in \mathscr{C}} t$$

contains precisely those things x such that Px is *true.*

Let us now shift gears, and look at what happens when a thing x does *not* have property P.

Let Qx be the predicate (*not* Px).

Qx is *true* precisely when it is *not* the situation that Px is *true*:

Qx is *true* for a given thing x when it is *not* the situation that:
for *every* s, s an element of $\mathscr{S}$ *guarantees* x is a member of $\neg s$.

Now we can apply *De Morgan Duality* to get:

Qx is *true* for a given thing x when it *is* the situation that:
there *exists* an s such that s an element of $\mathscr{S}$ and x is *not* a member of $\neg s$.

You should verify this. Hint: when does an implication fails?

But x is *not* a member of $\neg s$ exactly when it *is* a member of s, and therefore:

> Qx is *true* for a given thing x when it *is* the situation that:
> there *exists* an s such that s is an element of $\mathscr{S}$ and x is a member of s.

We have shown:

> The set
> $$\bigcup_{s \in \mathscr{S}} s$$
> contains precisely those things x such that Qx is *true*.

Recall that by *cancelling double negatives*, Px is equivalent to (*not not* Px). But (*not* Px) is just Qx.

Therefore Px is equivalent to (*not* Qx); i.e. Px is *true* exactly when Qx is *not true*.

And so:

> The set
> $$\bigcap_{t \in \mathscr{C}} t$$
> contains precisely those things x such that Qx is *not true*.

Putting everything together, we have discovered that:

1. The set
   $$\bigcap_{t \in \mathscr{C}} t$$
   contains precisely those things x such that Qx is *not true*.
2. The set
   $$\bigcup_{s \in \mathscr{S}} s$$
   contains precisely those things x such that Qx is *true*.

This result is summarized in the following theorem:

Theorem. (De Morgan Duality for Sets)
Let $\mathscr{S}$ be a non-empty collection of sets, and U a set such that every set in $\mathscr{S}$ is a subset of U. Then these two sets are the same

- $\bigcap_{s \in \mathscr{S}} \neg s = \bigcap_{t \in \mathscr{C}} t$
- $\neg \bigcup_{s \in \mathscr{S}} s$

In other words, the intersection of the complements is the complement of the union.
"If it's in every complement, then it's not in even one of the originals, and vice versa."

As an exercise, you should prove or at least convince yourself of this corollary.

Corollary. Let $\mathscr{S}$ be a non-empty collection of sets, and U a set such that every set in $\mathscr{S}$ is a subset of U. Then these two sets are the same

- $\neg \bigcap_{s \in \mathscr{S}} s$
- $\bigcup_{s \in \mathscr{S}} \neg s = \bigcup_{t \in \mathscr{C}} t$

In other words, the union of the complements is the complement of the intersection.
"If it's not in every one of the originals, then it's in at one of the complements, and vice versa."

A.4 Existentiality/Universality Distributive Behavior

Consider these two statements:

1. (For every dog y, y is a mammal) *OR* (Max the dog does not have fur)
2. For every dog y, (y is a mammal *OR* Max the dog does not have fur)

Mull this over for a second, and you can likely convince yourself that the first one is true if and only if the second one is true.

Next, consider these two statements:

1. (There exists a dog y where y does not bark) *AND* (Max the dog does not have fur)
2. There exists a dog y where (y does not bark *AND* Max the dog does not have fur)

Again, with a little thought, you can likely convince yourself that they are either both *true* or both *false*.

These equivalences are actually applications of the next theorem.

Theorem. Let P be a *property* and S a *statement*.
Then each of the following two statements is *true* if and only if the other one is:

1. For *every* x (Px is *true* *OR* S is *true*)
2. (Px is *true* for *every* x) *OR* (S is *true*)

In other words: "*OR* distributes over *universality*".
Also, each of the following two statements is *true* if and only if the other one is:

1. There is *at least one* x where (Px is *true* *AND* S is *true*)

2. (Px is *true* for *at least one* x) *AND* (S is *true*)

In other words: "*AND* distributes over *existentiality*".

Proof. Do this is an exercise. You could try using these tools:

- *Demonstrating universality.*
- *Applying universality.*
- *Demonstrating existentiality.*
- *Applying existentiality.*
- *Proof by Cases*
- *Process of Elimination*

□

A.5 Union/Intersection Distributive Behavior

Theorem. Let $\mathscr{S}$ be a non-empty collection of sets, and let B be a set. Then

$$\left(\bigcap_{s\in\mathscr{S}} s\right)\bigcup B = \bigcap_{s\in\mathscr{S}}\left(s\bigcup B\right)$$

Union with a single set "distributes over" an intersection of more than one set.

Proof. The theorem can be proven from scratch. By using the definitions of unions and intersections, *Proof by Cases*, and *Process of Elimination*, it is straightforward to show that the set on the left side is the same as the set on the right side. It's probably worth going through this procedure as an exercise.

But it turns out that what you'll do while in the above is quite similar to what you did in order to prove (as an exercise) the previous theorem about *AND/OR universality/existentiality distributive rules.*

It is more interesting to see if we can somehow instead just leverage that previous theorem.

Let us imagine any *arbitrary* thing. For the time being, keep this thing "fixed" in our minds, and let us refer to it using the symbol "t". In this context t is now this *specfic* fixed thing.

We would like to show that

$$t \in \left(\bigcap_{s \in \mathscr{S}} s\right) \bigcup B \quad \text{if and only if} \quad t \in \bigcap_{s \in \mathscr{S}} \left(s \bigcup B\right)$$

Let us proceed step by step.

$$\begin{aligned}
& t \in \left(\bigcap_{s \in \mathscr{S}} s\right) \bigcup B && \\
& && \text{i.f.f.} \quad \left[t \in \bigcap_{s \in \mathscr{S}} s\right] \text{ OR } \left[t \in B\right] \\
& \text{(definition of \textit{intersection})} && \text{i.f.f.} \quad \left[\text{for every } s,\ s \in \mathscr{S} \text{ implies } t \in s\right] \text{ OR } \left[t \in B\right] \\
& \text{(apply previous theorem here!)} && \text{i.f.f.} \quad \text{for every } s \left[\left(s \in \mathscr{S} \text{ implies } t \in s\right) \text{ OR } t \in B\right] \\
& \text{(meaning of \textit{``implies''})} && \text{i.f.f.} \quad \text{for every } s \left[\left(s \in \mathscr{S} \text{ is \textit{false} OR } t \in s\right) \text{ OR } t \in B\right] \\
& \text{(how \textit{``OR/existentiality''} works)} && \text{i.f.f.} \quad \text{for every } s \left[s \in \mathscr{S} \text{ is \textit{false} OR } \left(t \in s \text{ OR } t \in B\right)\right] \\
& \text{(definition of \textit{union})} && \text{i.f.f.} \quad \text{for every } s \left[s \in \mathscr{S} \text{ is \textit{false} OR } \left(t \in s \bigcup B\right)\right] \\
& \text{(meaning of \textit{``implies''})} && \text{i.f.f.} \quad \text{for every } s \left[s \in \mathscr{S} \text{ implies } \left(t \in s \bigcup B\right)\right] \\
& \text{(definition of \textit{intersection})} && \text{i.f.f.} \quad t \in \bigcap_{s \in \mathscr{S}} \left(s \bigcup B\right)
\end{aligned}$$

We have shown, for this *specific* thing t, that

$$t \in \left(\bigcap_{s \in \mathscr{S}} s\right) \bigcup B \quad \text{if and only if} \quad t \in \bigcap_{s \in \mathscr{S}} \left(s \bigcup B\right)$$

But "t" was a new symbol in this context, and the thing t was an *arbitrary* thing about which we used no extra information. This means that that we have actually proven a *universal* statement:

For *every* thing t,

$$t \in \left(\bigcap_{s \in \mathscr{S}} s\right) \bigcup B \quad \text{if and only if} \quad t \in \bigcap_{s \in \mathscr{S}} \left(s \bigcup B\right)$$

Recalling the *Extensionality* rule for *sets*,we can conclude that

$$\left(\bigcap_{s \in \mathscr{S}} s\right) \bigcup B = \bigcap_{s \in \mathscr{S}} \left(s \bigcup B\right)$$

what is what was to be proven.

□

Corollary. Let $\mathscr{S}$ be a non-empty collection of sets, and let B be a set. Then

$$\left(\bigcup_{s\in\mathscr{S}} s\right)\bigcap B = \bigcup_{s\in\mathscr{S}}\left(s\bigcap B\right)$$

Intersection with a single set "distributes over" a union of more than one set.

Note about the proof:

> Don't be alarmed if the equality in the very first step of the proof, which comes from the application the previous theorem, seems to be questionable. Although legitimate, it *is* definitely suspicious-looking, and there is some sleight of hand. What's going on is that we are actually applying the theorem to the collection of *complements* of sets in $\mathscr{S}$, but are then "indexing" those complements using the original sets that they came from.
>
> This is a kind of "abuse of notation", *relative to* the shorthand symbols/notation I've been using for the unions and intersection of collections of sets.
>
> Make sure you understand what's going on here, and that you can do the small amount of rework necessary to rewrite things so that they're completely rigorous.

Proof. Since we're proving a corollary, we should try to apply the previous theorem. The utility of *De Morgan Duality for Sets* will become apparent.

Apply the previous theorem to the complement of B and the complements of sets in $\mathscr{S}$, to obtain

$$\left(\bigcap_{s\in\mathscr{S}}(\neg s)\right)\bigcup(\neg B) = \bigcap_{s\in\mathscr{S}}\left((\neg s)\bigcup(\neg B)\right)$$

Take complements of both sides:

$$\neg\left[\left(\bigcap_{s\in\mathscr{S}}(\neg s)\right)\bigcup(\neg B)\right] = \neg\left[\bigcap_{s\in\mathscr{S}}\left((\neg s)\bigcup(\neg B)\right)\right]$$

Apply *De Morgan Duality for Sets* to both sides (the details are an exercise):

$$\left[\neg\bigcap_{s\in\mathscr{S}}(\neg s)\right]\bigcap\left[\neg(\neg B)\right] = \bigcup_{s\in\mathscr{S}}\left[\neg\left((\neg s)\bigcup(\neg B)\right)\right]$$

Apply *De Morgan Duality for Sets* to *portions* of each side (compare line above to line below):

$$\left[\bigcup_{s\in\mathscr{S}}\neg(\neg s)\right]\bigcap\left[\neg(\neg B)\right] = \bigcup_{s\in\mathscr{S}}\left[(\neg(\neg s))\bigcap(\neg(\neg B))\right]$$

Eliminate the "double complements ¬¬":

$$\left[\bigcup_{s\in\mathscr{S}} s\right]\bigcap B = \bigcup_{s\in\mathscr{S}}\left(s\bigcap B\right)$$

This completes the proof.

□

A.6 Statements with More Than One Universal and/or Existential

When you are studying continuity in a calculus class, you will likely see a statement like this:

> For *every* $\epsilon > 0$, and *every* b, there *exists* a $\delta > 0$, such that for *every* x, $|x - b| < \delta$ implies $|f(x) - f(b)| < \epsilon$.

Such statements, which involve multiple instances of universality, existentiality, or a combination of both, are common.

To understand the example above as intended by the mathematician who wrote it, you should think of it this way:

For *every* $\epsilon > 0$ it is true that:
{
For *every* b it is true that:
⟨
There *exists* a $\delta > 0$ such that:
(
For *every* x it is true that:
[
$|x - b| < \delta$ implies $|f(x) - f(b)| < \epsilon$
]
)
⟩
}

This way of grouping things where the smaller groups are towards the right side is called "right-associativity".

Important! This is just a convention for interpreting an Engligh sentence which involves multiple Existential or Universal quantifiers.

A.7 Disambiguating the Word "And"

In everyday English, the words "and" and "or" are sometimes used with different meanings than in math and logic.

For example, let's say someone says

> "Please go to the store and get apples *and* oranges."

Typically, they don't mean

> "Get things which are each individually and simultaneously *both* an apple *and* an orange."

Instead, they typically mean

> "Get apples. *In addition*, *also* get oranges."

This means the same thing as

> "Get apples *AND* get oranges."

In the previous sentence, "*AND*" is used in the way it is used in logic and mathematics.

Here's the upshot. In everyday English, the word "and" is sometimes used to conjoin together *more than* just the words immediately on either side of it. In the original sentence

> "Please go to the store and [get (apples *and* oranges)]"

"get" distributes over "apples *and* oranges".

The meaning is

> "Please go to the store and [(get apples) *and* (get oranges)]"

A.7.1 More about disambiguating the word "and"

Let's continue looking at the English sentence

> "Get apples and oranges."

This usually means

> "Get apples *AND* get oranges."

where the word "*AND*" is used as in logic and mathematics.

Imagine now that you are the person who has been instructed to do the shopping, and you are walking through the fruit section of the store. What are you to do?

- If you see a banana, then no action is required for it.

- If you see an apple, then get it.
- If you see a cherry, then no action is required for it.
- If you see an orange, then get it.

In short:

> "[If x is an apple then get x] *AND* [if x is an orange then get x]."

Let's now introduce some very standard notation and then work through things symbolically.

- Let Ax be the predicate "x is an apple".
- Let Ox be the predicate "x is an orange".
- Let Gx be the predicate "must get x".
- Let "$\rightarrow$" be shorthand for "*implies*".
- Let "$\wedge$" be shorthand for "*AND*".

In this notation, your shopping plan becomes the following predicate with placeholder x:

$$(Ax \rightarrow Gx) \wedge (Ox \rightarrow Gx)$$

You should convince yourself the predicate above means the is equivalent to:

$$(Ax \vee Ox) \rightarrow Gx$$

Translating back to English:

> "If x is either an apple *OR* an orange, then get x."

Remark: What we've just seen accounts for a typical kind of confusion which you'll experience if you are a business applications programmer who writes database queries based on requirements provided by end-users. The end-user will say "and", but then you will have to write a SQL "OR".

Exercise. Prove:

> For *every* x, the following are either both *true* or they are both *false*:
>
> - $(Ax \rightarrow Gx) \wedge (Ox \rightarrow Gx)$
> - $(Ax \vee Ox) \rightarrow Gx$

A.8 Disambiguating the Word "Or"

In daily life, the word "or" is used in two ways:

1. The *inclusive* sense. *"A or B"* means:
   "either A or B or both of them";
   "*at least one* of the two things".

2. The *exclusive* sense. *"A or B"* means:
   "one of A or B but *not* both of them";
   "*exactly one* of the two things".

In math and logic writing, "or" is *almost always* used in the *inclusive* sense. But in real life, you will frequently have to clarify the situation.

Suppose your friend says

> "Please go to the store and get me some apples *or* oranges."

What are they trying to convey to you? Will they be upset if you come back with *both* apples *and* oranges?

A.9 Disambiguating the Words "Unless" and "Except"

"Unless" and "except if" are like "or" in that they each have two different meanings.

1. The *weak* sense. *"A unless B" and "A except if B"* mean:
   - (not B) *implies* A
   - If B is *false* then *A* is *true*
   - It is *impossible* to have the combination (not A) and (not B)
2. The *strong* sense. *"A unless B" and "A except if B"* mean:
   - A is *true* if and only if B is *false*
   - A is *true* unless B is *true*, *in which case* A is *false*
   - A is *true* except if B is *true*, *in which case* A is *false*

It may feel counterintuitive, but in both *weak* and the *strong* sense "A unless B" and "A except if B" are *symmetric* in A and B:

> "A unless B" is *true* if and only if "B unless A" is *true.*

In math and logic, "unless" and "except if" are usually but not always used in the *weak* sense. Context and the author's intent matter.

In commmon English, it can be tricky to determine what someone means. Consider these shopping instructions:

> "Buy apples or maybe oranges, but just buy oranges unless apples are really cheap".

What are you supposed do if apples turn out to be really cheap? The instructions can easily be interpreted as either of these:

1. "I want apples *or maybe as a last resort* oranges, but just buy oranges unless apples are really cheap, in which case don't buy the oranges and just get the apples".
2. "I want apples *or maybe also oranges too*, but just buy oranges unless apples are really cheap, in which case buy *both* the apples and the oranges".

There are *at least* two additional factors in the instructions which make them problematic.

1. The ambiguous usage of "or" in "apples or maybe oranges".
2. The ambiguous usage of "just" in "just buy oranges" (compare with "buy just oranges"). The word "just" has multiple meanings. It can be used to emphasize something, but it can also be used to specify a limit or make something more precise.

Context matters!

B Names, Symbols, and Notation

Consider the items expressed in the following list:

names, words, designations, shorthand, notation, identifiers, references, handles

All of these are examples of *symbols*. A symbol is a thing – for example a shape, mark, sound, or gesture – which represents, refers to, or indicates something else.

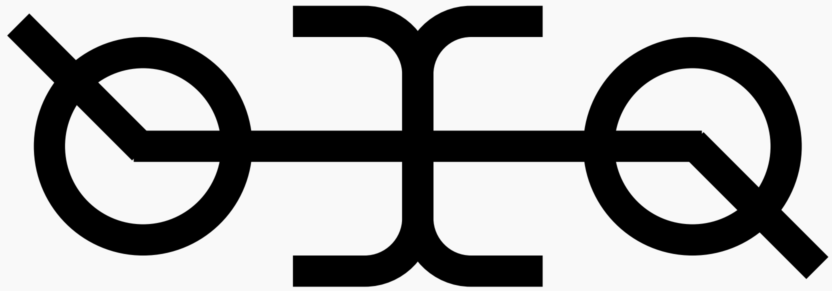
> *Note:*
> *What exactly trained philosophers or linguists consider symbols to be is way beyond the scope of these notes. But that's OK, because an informal and fuzzy understanding is good enough for what we need. If you are interested in digging deeper, then you can eneter the rabbit hole here:*
>
> - `https://en.wikipedia.org/wiki/Symbol`
> - `https://en.wikipedia.org/wiki/Referent`
> - `https://en.wikipedia.org/wiki/Philosophy_of_language#Reference`
> - `https://en.wikipedia.org/wiki/Lexical_analysis#Token`
> - `https://plato.stanford.edu/entries/reference/`
> - `https://en.wikipedia.org/wiki/Sign_(semiotics)`
> - `https://plato.stanford.edu/entries/peirce-semiotics/`

The thing indicated, referred to, or represented by a symbol is the symbol's *referent*.

The important thing for us is this:

> You *perceive* the *symbol* but then *interpret* the symbol as its *referent*.

Look at what's in the shaded box below:



That's a sweet shape, right? I like it; it sort of reminds me of a dancing alien. I'm quite happy that I figured out how to get LaTeX to draw it.

But now I'm going tell you something. I will use this shape to refer to the State of Michigan. So if you see me use this shape, you should think of the State of Michigan. The shape in the previous shaded box is now a *symbol* for the State of Michigan.

This means that you can make sense of what's in the next shaded box:

> One reason I like living in  is that there is so much water.

The upshot is this:

> The *shape that you saw* in the big box above is the *symbol*. It refers to the State of Michigan. The State of Michigan is the symbol's *referent*. When the symbol is used to convey meaning (such as when it's used in a sentence), it should be interpreted as the State of Michigan.

This is how symbols work in everyday language, but also in math. To understand something which has words or symbols that are new to you, it's important to make sure you're correctly understanding – and thinking in terms of – the symbols' referents.

B.1 Names for things vs. the things themselves

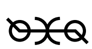
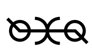
Let's consider what Alfred Tarski has to say on the difference between things and their names:

> It seems obvious, for instance, that the formula:
>
> $$3 = 2 + 1$$

is a true assertion, and yet some people are somewhat doubtful as to its truth. In their opinion, the formula appears to state that the symbols "3" and "$2+1$" are identical, which is obviously false since these symbols have entirely different shapes, and, therefore, it is not true that everything that may be said about one of these symbols may be said about the other.
...
In order to avoid doubts of this kind, it is well to make clear to oneself a very general and important principle upon which the useful employability of any language is dependent. According to this principle, whenever, in a sentence, we wish to say something about a certain thing, we have to use, in this sentence, not the thing itself but its name or designation.
...
The problem arises as to how we can set about to form names of words and expressions. There are various devices to this effect. The simplest one among them is ... quotation marks.

-Alfred Tarski,
Introduction to Logic and to the Methodology of Deductive Sciences

Here's my take on what Tarski is saying:

- A word is a symbol.
- A word is not the same thing as the meaning of the word.
  More generally, a symbol is not the same thing as its referent.
- Roughly, you *perceive* a *symbol* directly (by seeing or hearing, for example), but then *interpret* it as its referent to understand its intended meaning. A person using a symbol to communicate is expecting you to do this as you perceive then process their communication.
- A writer should wrap the *symbol* in quotation marks when they want to refer directly to that *symbol* instead of the symbol's *referent.*
- "my dog" is not the same thing as my dog.
- "  " is not the same thing as  .
- CORRECT: *My dog* is *Milo*.
- INCORRECT: *My dog* is *"Milo"*.
- CORRECT:  is the State of Michigan.
- INCORRECT: "  " is the State of Michigan.
- CORRECT: *The name* of my dog is *"Milo"*.
- INCORRECT: *The name* of my dog is *Milo*.

- CORRECT: “  ” is a symbol for the State of Michigan.
- INCORRECT:  is a symbol for the State of Michigan.

This convention, whereby when writing sentences we always refer to a thing by using a symbol which has that thing as its referent, is great and unambiguous. All the same, it can still be tricky. In particular, it can be awkward to write sentences which express things *about* symbols. You must then use a *symbol for the symbol you want to say something about*.

> Young man, in mathematics you don't understand things. You just get used to them.
> -John von Neumann

Fortunately, there are ways to express ideas about symbols which don't require the quoting convention. One such way is given by the following two-step procedure:

1. First, present an isolated and out-of-context instance of the *symbol* in such a way that a perceiver is not inclined to *interpret* the symbol as its *referent*.
   Specifically, don't use the symbol in a sentence.

2. Second, comment on this presentation in such a way that you achieve your desired communication about the *symbol*.

(There is a helpful example of this on the next page.)

=== BEGIN HELPFUL EXAMPLE ============================

In what follows, the mark in Box 1 can be changed to anything, as long as what's in the subsequent boxes is changed accordingly. Also, its referent (in this case the State of Michigan) can be changed to anything.

Box 1

The shape you see in Box 1 *is a symbol for* the State of Michigan.
The shape you see in Box 1 *is a name for* the State of Michigan.
The shape you see in Box 1 *refers to* the State of Michigan.
In writing, the symbol in Box 1
is interpreted as the State of Michigan itself.

Box 2

“  ”

The shape you see in Box 2 *is a name for* the object in Box 1.
The shape you see in Box 2 *is a symbol for* the object in Box 1.
The shape you see in Box 2 *refers to* the symbol in Box 1.
In writing, the symbol in Box 2 *is interpreted as* the symbol in Box 1.

Box 3

““  ””

The shape you see in Box 3 *is a name for* the object in Box 2.
The shape you see in Box 3 *is a symbol for* the object in Box 2.
The shape you see in Box 3 *refers to* the symbol in Box 2.
In writing, the symbol in Box 3 *is interpreted as* the symbol in Box 2.

Box 4

“““  ”””

The shape you see in Box 4 *is a name for* the object in Box 3.
The shape you see in Box 4 *is a symbol for* the object in Box 3.
The shape you see in Box 4 *refers to* the symbol in Box 3.
In writing, the symbol in Box 4 *is interpreted as* the symbol in Box 3.

Etc etc etc . . .

Etc etc etc . . .

Etc etc etc . . .

=== END HELPFUL EXAMPLE ============================

Let's now use the HELPFUL EXAMPLE to revist and understand some of the items in the list which followed the above Tarski quote. Recall that when writing sentences, the *names/symbols* for things must be to refer to those things.

- This sentence is GOOD:

  >  is the State of Michigan.

  In HELPFUL EXAMPLE, we saw the symbol used here appear in Box 1, and saw

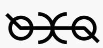
  > In writing, the symbol in Box 1 *is interpreted as* the State of Michigan itself.

  So the sentence can be converted to:

  > The State of Michigan ***is*** the State of Michigan.

  This makes sense.

- This sentence is BAD:

  > "  " is the State of Michigan.

  In HELPFUL EXAMPLE, we saw the symbol used here appear in Box 2, and saw

  > In writing, the symbol in Box 2 *is interpreted as* the symbol in Box 1.

  So the sentence can be converted to:

  > The symbol in Box 1 ***is*** the State of Michigan.

  This does not make sense. The symbol in Box 1 is a symbol *for* the State of Michigan, but is *not the same thing* as the State of Michigan itself.

- This sentence is GOOD:

  > "  " is a symbol for the State of Michigan.

  In HELPFUL EXAMPLE, we saw the symbol used here appear in Box 2, and saw

  > In writing, the symbol in Box 2 *is interpreted as* the symbol in Box 1.

  So the sentence can be converted to:

  > The symbol in Box 1 ***is a symbol for*** the State of Michigan.

  This makes sense.

- This sentence is BAD:

  >  is a symbol for the State of Michigan.

  In HELPFUL EXAMPLE, we saw the symbol used here appear in Box 1, and saw

  > In writing, the symbol in Box 1 *is interpreted as* the State of Michigan.

  So the sentence can be converted to:

  > The State of Michigan itself ***is a symbol for*** the State of Michigan.

  This does not make sense, becasue a symbol for the State of Michigan is not the same thing as the State of Michigan.

So that's the gist of it.

Now let's look at a more complicated example.

> """"  """"
> is
> *a symbol for a symbol for a symbol for the State of Michigan.*

Is this sentence true? We have to work it out.

> In HELPFUL EXAMPLE, we saw the symbol used here appear in Box 4, and saw:
>
> In writing, the symbol in Box 4 *is interpreted as* the symbol in Box 3.
>
> Using this,
>
> """"  """"
> is
> *a symbol for a symbol for a symbol for the State of Michigan.*
>
> can be inpterpreted as
>
> *The symbol in Box 3*
> is
> *a symbol for a symbol for a symbol for the State of Michigan.*
>
> The resulting sentence now has no symbols, but we don't know yet if it's true. However, we saw:

The shape you see in Box 3 *is a symbol for* the object in Box 2.

which just means that the symbol in Box 3 *is a symbol for* the symbol in Box 2. So we get

A symbol for the symbol in Box 2
is
a symbol for a symbol for a symbol for the State of Michigan.

Similarly:

The shape you see in Box 2 *is a symbol for* the object in Box 1.

So the symbol in Box 2 is a symbol for the symbol in Box 1, and we get

A symbol for a symbol for the symbol in Box 1
is
a symbol for a symbol for a symbol for the State of Michigan.

Finally, the symbol in Box 1 is a symbol for the State of Michigan, and we arrive at

A symbol for a symbol for a symbol for the State of Michigan
is
a symbol for a symbol for a symbol for the State of Michigan.

So it worked out, and we have a true statement!

So this has been a very long digression. To end it, I will explain why something you might be tempted to do actually ***DOES NOT WORK THE WAY YOU HOPE IT WILL***.

In the previous example, we started off by doing converting

“““  ”””
is
a symbol for a symbol for a symbol for the State of Michigan.

into

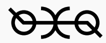
The symbol in Box 3
is
a symbol for a symbol for a symbol for the State of Michigan.

Why did we have to write out “*The symbol in Box 3*” instead of just actually substituting in the shape that appears in Box 3?

Box 3

““  ””

If we did so, then we'd get

““  ””
is
a symbol for a symbol for a symbol for the State of Michigan.

This is incorrect, and does not mean the same thing as what we started with! It's important to always remember that symbols used in sentences are not to be interpreted as *themselves*, but rather what they refer to. Keeping this in mind, we see that

““  ””
is
a symbol for a symbol for a symbol for the State of Michigan.

means

The symbol in Box 2
is
a symbol for a symbol for a symbol for the State of Michigan.

B.2 Additional examples

Example:

Joe is a cat that I have.
Joe *herself* has these properties, among others:

- black
- fuzzy
- squishy
- smells nice
- weighs 8 lbs

On the other hand, the *name* I used for her in the sentence

“Joe is a cat that I have”

has these properties, among others:

- consists of three letters in order: “J”, “O”, and “E”
- rhymes with “Ohh” when spoken

- does not have a weight associated with it

Example:

- I have a cat, and the fourth word in this sentence *indicates* something with whiskers.
- I have a cat, and the *fourth word* in this sentence *is* "cat".
- I have a cat, and *the name of* the fourth word in this sentence *is* ""cat"".

Example:

> On one hand,
>
> $$3 = 2 + 1$$
>
> is *true*. The statement involves *numbers* referred to by *names* "3", "2", and "1".
> One of the involved numbers is 3, and its name is "3".
> On the other hand, the *symbol/notation* I used in order to express that truth is
>
> "$3 = 2 + 1$"
>
> In fact, that symbol/notation actually consists of five individual smaller symbols:
> "3", "=", "2", "+", and "1".

B.3 Defining new names, words, symbols, shorthand, and notation

What we're going to about talk about here applies not only to names, but also all of these:

> *names, words, designations, shorthand, notation, identifiers, references, handles, symbols*

Generally speaking, people invent and use names for things when indicating and communicating about them in other ways – long descriptions, drawing ad hoc one-off pictures, odd sounds or strange gestures, etc. – becomes too repetitive or tedious.

Let's look at an actual math example. This is from p. 10 of
Principles of Mathematical Analysis (Third Edition), by Walter Rudin:

> Rudin gives us three sentences (in the gray boxes below) telling us what is is up to. I will explain how I make sense of them, one by one.
>
> **Sentence #1:**
>
> > We shall now prove the existence of n-th roots of positive reals.
>
> My interpretation of sentence #1:

We're about to talk about something which we will ***call or refer to*** as **"*n*-th roots"**. In fact, we're about to say what it means for something ***to be*** an n-th root, and show that these so-called **"*n*-th roots"** exists in some way which will be made more precise in the nect sentence.

Sentence #2:

> **1.21 Theorem** For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.

My interpretation of sentence #2:

This says two things.
First, it tells you what it ***means*** for something (here generically identified by "y") ***to be*** an n-th root of something else (here generically identified by "x"). A number y being an n-th root of a number x is a particular relationship between the three numbers n, x, and y, where the order of the three matters.
Second, it says that – given the specified conditions – a thing has exactly one n-th root.

Sentence #3:

> This number y is written $\sqrt[n]{x}$ or $x^{1/n}$.

My interpretation of sentence #3:

He is telling you that he is introducing ***new shorthand/notation*** to form ***new symbols*** – "$\sqrt[n]{x}$" and "$x^{1/n}$" – from a given symbol "x". The new symbols "$\sqrt[n]{x}$" and "$x^{1/n}$" *identify* the n-th root of the number indicated by symbol "x". This means that if you see one of those symbols, you should think in your head:
"the unique number y which has the property that $y^n = x$".

Here is a crucial point:
Actually, Rudin is telling us how the <u>interpretation</u> of the symbols "$\sqrt[n]{x}$" and "$x^{1/n}$" relates to the <u>interpretation</u> of the symbol "x". He's informally and implicitly giving us a kind of pattern, which allows us to relate the interpretations of two symbols when they match the pattern. And so using the new notation is <u>not limited</u> to things identified by the symbol "x" and "n". The symbols "x" and "n" can be replaced by any other symbols and we are allowed to make sense of the resulting symbols in the obvious way. This makes the new notation

much more useful than if it were limited to the symbols "x", "n", and "$x^{1/n}$"!

For example:

- *The symbol "$\sqrt[n]{a}$" indicates the n-th root of whatever "a" indicates, so that $\sqrt[n]{a}$ is the n-th root of a*
- *The symbol "$z^{1/7}$" indicates the 7-th root of whatever "z" indicates, so that $z^{1/7}$ is the n-th root of 7-th root of z.*
- *The symbol "$\sqrt[k]{16}$" indicates the k-th root of 16, if we already know that the symbol "k" indicates a positive integer.*
- *The symbol "$16^{1/4}$" also indicates the 4-th root of 16, so that $16^{1/4} = 2$.*

You probably noticed that Rudin does not abide by Tarski's quoting convention mentioned earlier. While this is unfortunate, it is also ubiquitous in math written by non-logicians. Tarski is a logician, but Rudin is an analyst. They come at things from different places. The author leaves it to you to interpret things with precision, either because they think things are clear enough as they stand, or they don't want to derail the flow of the text by being too pedantic.

The moral of the story is that it's up to you to make sure you know what an author is trying to communicate.

C More on Composition of Relations and Functions

Let P and Q be relations, and recall the definition of the *composite* relation $Q \circ P$:

$$Q \circ P = \{(x, y) : \text{ there exists a } t \text{ such that } (x, t) \in P \text{ and } (t, y) \in Q\}$$

Exercise: Equivalent Definition of Composition

Verify that $(x, y) \in Q \circ P$ if and only if there exists an "*finite sequence*" $\langle v_1, v_2, v_3 \rangle$ of three values v_1, v_2, and v_3 such that

$$\begin{aligned} &v_1 = x, \\ &(v_1, v_2) \in P, (v_2, v_3) \in Q, \\ &\text{and } v_3 = y. \end{aligned}$$

If what it means to verify this is not clear to you, then review these items:

- **Applying/using existentiality**
- **Demonstrating/showing/proving existentiality**
- **A is true if and only if B is true**

Note: A finite sequence (also called an "n-tuple") is just like an ordered pair, except there can be more than two components. The key property of finite sequences is that two of them are the same if and only if they have the same length and the same values in corresponding positions.

Generalized Definition of Composition

Let $n \geq 2$ be an integer, and let R_1, R_2, ..., R_n be relations. We are about to *define* a new symbol.

Let the symbol " $\bigodot_{k=1}^{n} R_k$ " refer to the relation which is the set of all ordered pairs (x, y) such that there exists a finite sequence $\langle v_1, v_2, \ldots, v_n, v_{n+1} \rangle$ of $n+1$ values with

$$\begin{aligned} &v_1 = x, \\ &(v_1, v_2) \in R_1, (v_2, v_3) \in R_2, \ldots, (v_n, v_{n+1}) \in R_n, \\ &\text{and } v_{n+1} = y. \end{aligned}$$

In other words

$$\bigodot_{k=1}^{n} R_k = \Big\{ (x, y) : \text{ there exists an } (n+1)\text{-tuple } \langle v_1, v_2, \ldots, v_n, v_{n+1} \rangle \text{ with}$$
$$v_1 = x, (v_1, v_2) \in R_1, (v_2, v_3) \in R_2, \ldots, (v_n, v_{n+1}) \in R_n, v_{n+1} = y \Big\}$$

For example, if $n = 2$,

$$\bigodot_{k=1}^{2} R_k = \Big\{ (x, y) : \text{ there exists an 3-tuple } \langle v_1, v_2, v_3 \rangle \text{ with}$$
$$v_1 = x, (v_1, v_2) \in R_1, (v_2, v_3) \in R_2, v_3 = y \Big\}$$

Thus $\bigodot_{k=1}^{2} R_k = R_2 \circ R_1$, by *Exercise: Equivalent Definition of Composition.* For $n = 4$, we have

$$\bigodot_{k=1}^{4} R_k = \Big\{ (x, y) : \text{ there exists an 5-tuple } \langle v_1, v_2, v_3, v_4, v_5 \rangle \text{ with}$$
$$v_1 = x, (v_1, v_2) \in R_1, (v_2, v_3) \in R_2, (v_3, v_4) \in R_3, (v_4, v_5) \in R_4, v_5 = y \Big\}$$

And so on.

Exercise: Very Useful Facts About Composition of Relations and Functions

You will likely be able – maybe with some effort – to work your way through the following:

Comp-1 *(Composition of Relations is "Associative")*
Let P, Q, and R be relations. Let $S = Q \circ P$ and $T = R \circ Q$.
Prove that $R \circ S = T \circ P$.
In other words, prove that $R \circ (Q \circ P) = (R \circ Q) \circ P$.

Hint: You must show $R \circ S \subset T \circ P$ and $T \circ P \subset R \circ S$. You will need to use the definition of composition, ***Applying/using existentiality*** *(several times), and* ***Demonstrating/showing/proving existentiality*** *(several times). To start off, suppose that $(a, b) \in R \circ S$. Then there exists some x_1 such that $(a, x_1) \in S$ and $(x_1, b) \in R$. But $S = Q \circ P$, so there exists some x_2 such that $(a, x_2) \in P$ and $(x_2, x_1) \in Q$. Notice this means that $(x_2, b) \in R \circ Q$. Proceed.*

Comp-2 Let $n \geq 2$ be an integer, and let R_1, R_2, ..., R_n be relations.
Verify that

$$R_n \circ (R_{n-1} \circ (\cdots \circ (R_2 \circ R_1) \cdots)) = \bigodot_{k=1}^{n} R_k$$

Hint: Use induction and Exercise: Equivalent Definition of Composition.

Comp-3 Let $n \geq 2$ be an integer, and let F_1, F_2, ..., F_n be functions.
Verify that the finite sequence $< v_1, v_2, \ldots, v_n, v_{n+1} >$ of $n + 1$ required by the definition of

$$(x, y) \in \bigodot_{k=1}^{n} F_k$$

is actually *unique.*

Hint: Suppose $< w_1, w_2, \ldots, w_n, w_{n+1} >$ is some (potentially different) sequence which also demonstrates $(x, y) \in \bigodot_{k=1}^{n} F_k$. Show that the two sequences are in fact the same by showing that $w_k = v_k$ for each $k = 1, 2, 3, \ldots n + 1$.

Comp-4 Let F_1, F_2, ..., F_n be functions. Verify that the domain of the composite function $\bigodot_{j=1}^{n} F_j$ is contained in the domain of F_1, and they are equal if and only if for each $k = 1, 2, \ldots, n - 1$ it's true that the range of $\bigodot_{j=1}^{k} F_j$ is contained in the domain of F_{k+1}.

Hint: Try induction.

Comp-5 Verify the following special case of *Comp-4*.
The domain of

$$F_n \circ (F_{n-1} \circ (\cdots \circ (F_2 \circ F_1) \cdots)) = \bigodot_{j=1}^{n} F_j$$

equals the domain of F_1 if

$$\begin{aligned} &\text{range of } F_1 \subset \text{ domain of } F_2 \\ &\text{range of } F_2 \subset \text{ domain of } F_3 \\ &\ldots \text{ etc } \ldots \\ &\text{range of } F_{n-1} \subset \text{ domain of } F_n \end{aligned}$$

Comp-6 (The Composite of Functions is a Function)

Verify that if F_1, F_2, ..., F_n are functions, then $\bigodot_{k=1}^{n} F_k$ is a function.

!!! Spoiler alert !!!

Here's a pretty complete sketch of one proof:

Let $(x, y) \in \bigodot_{k=1}^{n} F_k$ and $(x, z) \in \bigodot_{k=1}^{n} F_k$.
From the Generalized Definition of Composition,
there is a sequence $< v_1, v_2, \ldots, v_n, v_{n+1} >$ such that

$$v_1 = x, (v_1, v_2) \in F_1, (v_2, v_3) \in F_2, \ldots, (v_n, v_{n+1}) \in F_n, v_{n+1} = y$$

Similarly, there is a sequence $< w_1, w_2, \ldots, w_n, w_{n+1} >$ such that

$$w_1 = x, (w_1, w_2) \in F_1, (w_2, w_3) \in F_2, \ldots, (w_n, w_{n+1}) \in F_n, w_{n+1} = z$$

Since F_1 is a function and $v_1 = x = w_1$, we see that $v_2 = w_2$.
Since F_2 is a function and $v_2 = w_2$, we see that $v_3 = w_3$.
Continuing in this way (or using induction to be rigorous), we conclude that $v_{n+1} = w_{n+1}$, and this shows that $y = z$.

C.1 Calculating a Composite of Functions In Practice

Let F_1, F_2, ..., F_n be functions as in *Comp-5* where the range of one function is contained in the domain of the next. Then from *Comp-5*, the domain of $\bigodot_{j=1}^{n} F_j$ is the same as the domain of F_1.

Suppose now that someone provides us with a particular element x in the domain of F_1. Then (by *Comp-5*) x is in the domain of $\bigodot_{j=1}^{n} F_j$. So we know that (by *Comp-6*) there is a unique y which satisfies

$$(x, y) \in \bigodot_{j=1}^{n} F_j$$

and that (by *Comp-3*) there is a unique sequence $< v_1, v_2, \ldots, v_n, v_{n+1} >$ such that

$$v_1 = x, (v_1, v_2) \in F_1, (v_2, v_3) \in F_2, \ldots, (v_n, v_{n+1}) \in F_n, v_{n+1} = y$$

This Generalized Definition of Composition we've been working with so far is very useful in abstract, pure math settings. But in practical applications (e.g. engineering) where each of the functions F_j can be calculated explicitly, people generally do not think about composing functions in this way. Instead, it's typical to think of composing functions as a step-by-step process.

If h and g are functions where the range of g is contained in the domain of h, then it makes sense to compose h with g to get a function $h \circ g$.

(Recall the "$f(s)$" notation for functions, used to indicate the unique value associated with s by the function f. That is, $f(s)$ is the unique value t such that $(s,t) \in f$. Similarly, the notation "$\left(\bigodot_{j=1}^{n} F_j\right)(s)$" is used for the function $\bigodot_{j=1}^{n} F_j$.)

Let x be in the domain of $h \circ g$. The symbol "$(h \circ g)(x)$" is frequently used to indicate the unique value y such that $(x,y) \in h \circ g$. In other words, $(x,(h \circ g)(x)) \in h \circ g$.

In addition, however, you will quite frequently encounter the symbol "h(g(x))".

Question: How to interpret the expression "h(g(x))"?

- TODO

The answer – thankfully! – is that it doesn't matter in real life. All of the above interpretations yield the same function, namely $h \circ g$.

Let's look at the situation and see what's going on.

Here is how you'd calculate $h(g(x))$, or more generally an expression of the form "$F_n(F_{n-1}(\cdots(F_2(F_1(x)))\cdots))$", if you work your way from the inside out:

"Inside Out" Calculation Of $F_n(F_{n-1}(\cdots(F_2(F_1(x)))\cdots))$

1. TODO

On the other hand, here is how you'd calculate $h(g(x))$, or more generally an expression of the form "$F_n(F_{n-1}(\cdots(F_2(F_1(x)))\cdots))$", if you work your way from the outside in:

"Recursive" Calculation Of $F_n(F_{n-1}(\cdots(F_2(F_1(x)))\cdots))$

1. TODO

C.1.1 The above three ways of calculating a composite function agree, at least in real life.

You can breathe a sigh of relief.

Theorem. *The above three ways of calculating a composite function agree, at least in real life.*

Let $F_1,\ F_2,\ \ldots,\ F_n$ be functions, and let x be any element in the domain of the composite function $\bigodot_{j=1}^{n} F_j$. Then the following three values for y are the same:

1. $y = \left(\bigodot_{j=1}^{n} F_j\right)(x)$ as determined by the <u>*Generalized Definition of Composition*</u>.
   That is, the unique value y such that there exists a sequence $\langle v_1, v_2, \ldots, v_n, v_{n+1}\rangle$ of $n+1$ values such that
   $v_1 = x,\ (v_1, v_2) \in F_1,\ (v_2, v_3) \in F_2,\ \ldots,\ (v_n, v_{n+1}) \in F_n,$ and $v_{n+1} = y$.
2. $y = F_n(F_{n-1}(\cdots(F_2(F_1(x)))\cdots))$ as determined by the <u>*"Inside Out" Calculation*</u>.
3. $y = F_n(F_{n-1}(\cdots(F_2(F_1(x)))\cdots))$ as determined by the <u>*"Recursive" Calculation*</u>.

Proof.

TODO

□

TODO:

- Application to understanding how to actually calculate things like

$$\frac{42a^2 + b}{100}$$

  It is what's associated by a composition of four functions with the following input 5-tuple:

$$(42, a, 2, b, 100)$$

- Mention Dummit & Foote p.19 proof of generalized associative law.

D TODO: Find a home for this stuff

<u>*BRYAN – figure out how to draw a nice boundary so that the approach cannot be faulted.*</u>

TODO – Make this document bulletproof.

<u>*How do "algorithms" fit into math?*</u>

TODO – What the heck does this even mean?
??? I think you might be able to use the recursion theorem to justify thinking of the each-step-outputs of algorithms as a sequence?

<u>*Note About Ellipses("Dots") and Using Parentheses And Brackets For "Grouping"*</u>

TODO – What the heck does this even mean?
Should this be moved to the *Elementary Concepts* section?
Parentheses Avoiding ambiguity?
Dots indicating what you should imagine doing were you to perform some procedure for an actual natural number?

More about symbols

Implicit “for all” quantifier when definitng notation??? “Composing” symbols???

What are mathematical objects

TODO – What the heck does this even mean?
Answer - it doesn’t matter which philosophy you have.

In practice, regardless of what they believe privately, many mathematicians seem to treat mathematical entites as if they are actual (but abstract) entites (`https://en.wikipedia.org/wiki/Abstract_particulars`). It often seems, when reading mathematics, that a `https://iep.utm.edu/mathplat/` perspective is being adopted.
For example: “Let B be the smallest integers such that . . . ”.
It doesn’t really matter if integers are *real* – in the game of math, you use them as if they are.
Handles/variables for math entites.
Again: “Let B be the smallest integers such that . . . ”, let $A = 42$.
Now you can use these to build something new: “Let $C = \{\mathrm{A}, \mathrm{B}\}$”.

Symbols/handles/variables can be thought of as referring to things in “abstract value space” – some unchanging mathematical universe. “Using” a math entity does not change it: you’re not going to break it by playing with it.

When a mathematician says in some context “Let x be <<<some object>>>”, they are giving themselves and us license to believe (or pretend to believe) we have such an object in our minds eye, and that it will persist there and be accessible by “x”, until we no longer need it, we are no longer in that context, or we use “x” for something else in that same context.

Kleene’s 3 level perspective

TODO – cannot espace the informal human reasoning. Human-level primitives/reasoning is basis for informal math and reasoning *abut* formal languages.

Real-life calculations vs functions as sets of ordered paris

TODO – Real-life to informal axiomatic math isomoprhism???

Recursion theorem

TODO – Real-life to informal axiomatic math isomoprhism??? Some people take this as an axiom, or as an obvious application of induction... but be careful.
Mention Enderton’s “fake proof”.

Hidden inductive definitions

TODO – for example the function $x \mapsto x^n$... se Munkres p.35.
Definitions using ellipses and parentheses vs inductive definitions.

Composition of functions TODO

Outside-in calc: From the persepctive of processing symbols (or a composition of *symbols*).
Inside-out calc: From the perspective of parentheses as groupings.

Can using handles and considering entites as abstract particulars help out reconciling all definitions/interpretations?

Euclidean algorithm

TODO – use recursion theorem

More about informal human reasoning

TODO
It's primitive, and we are always functioning at this level, but... you can train yourself, and you can check your work!
We can write down the steps of our reasoning, and verify that they make sense, and don't violate the guildelines qwe set for ourselves.
What those guidelines are and how stringent they are varies from person to person, but there is a general consensus about the most important ones.
Also, these guidelines can be expressed symbolically, and as such are the starting point for developing the string expressions found in sy7mbolic (formal) logic.

Give a good, actual math example.

E FIXME::TODO

I need a redo for this whole document.

After putting this together, I realized that it has way to much stuff in it. What I should do is completely trim it down, and retain only on what is necessary to understand these:

- Enderton's *Elements of Set Theory* book
- Velleman's *How to Prove It* book
- Beck's *The Art of Proof* book
- Munkres' *Topology* book
- Leon Henkin's article *On Mathematical Induction* `https://www.jstor.org/stable/2308975`

There's no fun in informal mathematics (which includes pure, abstract math) if you can't imagine the things you're playing with as being real.

E.1 Velleman's ***How to Prove It*** book

- p.8 Basic reading ability, which requires short (ordered) lists of words and sentences.
- p.8 Introducing basic mathematic deductive reasoning.
- p.8 Statements, truth values, truth, falsity.
  CRUCIAL: Human-level statements, truth, falsity.
- p.8 Argument = sequence of steps in deductive reasoning.
  CRUCIAL: Requires intuitive understanding of small natural numbers to get short (ordered) list of steps in the argument.
- p.8 **CRUCIAL:** Section starting with "But is this conclusion really correct".
  Applying human-level reasoning to check validity of a deductive argument.
  Looking at the deductive argument and involved statements as if they were a things/objects/situations, and then reasoning about it "from above"/"observer's vantage point".
  "In this case, ...": Imaginary scenario human reasoning.
- p.8 Process of elimination and implication elimination inside deductive reasoning, vs applying human-level elimination and implication elimination to check the validity.
- p.9 *Defining* the word *"valid"* as a *property* of an argument.
- p.9 Saying that something is *not valid* because if does not have the required propoerty to be valid.
- p.9 Small natural numbers.
- p.9 Things like "P" and "Q" "stand for" "statements".
  CRUCIAL: This is completely informal.
- p.9 Informal notion of the *form* of an argument.

- p.11 Parentheses. Saying it's similar to parenetheses in algebra.
- p.12 strings = finite sequences.
  Translating human-readable argument into string of some form is *primitive.*
- p.12 meaningless
- p.15 Truth tables to capture possibilities.
  But reason about the truth tables by using human-level elimination and implication.
- p.15 Making a truth table requires being able to list out the members of $\{true, false\}^n$, and then fill in values for the calculated colums.
- p.17 Mentally parsing simple logical formulas.
- p.17 Using truth table to show argument is valid.

- p.19 Examining or reasoning with truth tables requires basic human-level counting and human-level reasoning (if-then, at least one, both, all, exists, etc...).
  Also (p.16-p.19), filling out a truth table or reasoning out the truth value of a composite statement is a human-level step-by-step process.
  CRUCIAL: This means human-level step-by-step processes are primitive.

- p.23 It appears to me from his usage of parentheses in formulas and the way he is constructing truth tables, that parsing out information from parenthesized human-level expressions is supposed to mentally happen in a step-by-step way using some primitive mental framework which has something like expression trees.

- p.26 Variables are symbols, and symbols ”represent” things.

- p.28 collection, elements, explicitly specifying sets using braces, set-builder notation.

- p.29 Reasoning and drawing conclusions about sets appears to involve thinking about them the same way as if they were actually tangible, real-life objects.
  You are allowed to fix a mathematical entity in your mind and then reason about it in a step-by-step human way.

- p.31 Definition of truth set is using the subset specification axiom.

- p.32 Assume we have these sets of numbers, can reason about them as if they were real tangible objects and everyone agrees about what precisely they are.

- *context, universe of discourse, range over*

- p.35 He is assuming that we know how symbols/notation/shorthand work at a human level in order to define symbols for intersection and union of sets.
  Note that a lot of times, the meaning of a new symbol is definied in terms of other symbols, and the meaning of new notation is definied in terms of other notation.

- p.58 human-level every, at least one, and definition of quantifiers in terms of these.

- p.173 Ordered pairs is primitive for the author.

- p.234 Look very closely at this proof, especially the reasoning at the end for
  $(g \circ f)(a) = c = g(b) = g(f(a))$.

- p.293-294 Informal definition of recursion and recursively defined functions.
  No attempt to formally justify.
  Says basically ”because we can compute $f(n)$ for any n, we really do have a function.”
  Basically taking it as an axiom that this works and that you really get a function defined on the totality of the natural numbers. This is better than in Beck’s book p.35, where there is a bullshit fake proofof recursion given, the exact type of proof that Enderton calls out in his set theory book on p. 76.

- p.294 Definitions with ellipses (“. . . ”) are often secretly recursive definitions.

- p.326-327 Euclidean Algorithm for finding gcd(m, n).
- p.328 The Euclidean Algorithm is used in a proof that gcd() is a linear combo of arguments.
- p.372 Using the natural numbers to define *finite*.
- p.383 Definition of finite sequence as a function on $\{1, 2, 3, \ldots, n\}$.

E.2 Beck's *The Art of Proof* book

- p.xvi "assumed to know" something" ?=? "assumed to have a working understanding of how more experienced mathematicians think of" something ???
- p.xvi Understanding of mathematical "truth" has evolved over a period of two thousand years?
- p.4 set is primitive.
- p.4 A "distinguished" set of integers satisfying some axioms is taken as primitive.
- p.4 Parentheses determining "order of operations" interpreted as "do what's on the inside first".
  I think this means that step-by-step calculations are considered primitive.
- p.5 "same number" ===¿¿¿ thinking of math entities in the way you would think of more concrete and real-life objects?
- p.5 Punting of "truth".
  Interpreting the "truth" of a statement in a particular context as one which can be used as a basis for further deductions.
- p.6 A "proof" is a list/sequence of statements where each statement follows logically from previous ones.
- p.7 properties and predicates are primitive, and informal.
- p.10 Defining new binary operation (subtraction) and notation for that operation.
- p.11-12 There is an "unexamined basis of logic" which works at the human level, and is required to even make the kinds of philosophical arguments about "what is valid logic" (resulting in the "examined logic") that people have been wrestling about for 2000 years.
- p.12 The human-level "unexamined basis of logic" natural way of thinking is good enough, but you have to be trained enough to know what other peple would generally consider to be valid, and what the general "rules of the game" are.
- p.12 Not possible to state all of the axioms of thought or the universe that we are assuming in order to do math.

- p.12 Distinction between (A) logic/set theory, and (B) math based on logic and set theory.
  Necessary compromise in order to not get stuck on philosophy:
  We be intuitive and informal about the foundations of math from (A) logic/set theory.
  But, to then proceed to build the structure of mathematics, we have to then rigorously apply the democratically accepted logical thought processes available as basic human capabilities.

- (NOTE FROM BRYAN) I think that there is a good analogy between musical training and mathematical training.
  What counts as "generally acceptable, logical, basic human-level reasoning" is something learned by repetition, training, and pattern recognition.
  This feels similar to how you learn to play an instrument or where musical phrases are going.
  If this is the case, then maybe it's OK to think of it as an intuitive thing which you can enjoy and not stress out about too much.

- p.15 Assuming Excluded Middle as a basis from Proof By Contradiction.

- p.18 Assuming Induction Axiom for the naturals.

- p.18 Again, assuming that there is a (informal) primitive notion of "properties".

- p.25 These are informal and primitive: for all, there exists, and, or, if . . . then (guarantees), impossible, not, etc.

- p.26 quantifiers

- p.31 In math, there are always going to be undefined words which have only intuitive meanings (for example "collection" or "property").

- p.35 Baby Recursion Theorem.
  CRUCIAL: THE PROOF IS BULLSHIT. ENDERTON SET THEORY CALLS OUT FAKE PROOFS OF THE RECURSION THEOREM ON p.76.
  CRUCIAL: Alternatively, you could look at it from the perspective that a mathematician might think that the proof is "good enough".

- p.52 "ordered pair" is primitive.

- p.63 "unique up to (whatever)" needs explanation.

- p.69 Algorithm is primitive and informal. A step-by-step procedure which is obviously unambiguous and mechanical.

- p.70 Elementary School Addition Algorithm:
  CRUCIAL: How to interpret this is super important!!! **WHY IS IT ALLOWED TO THINK OF THE OUTPUTS OF THE ALGORITHM AS INTEGERS???**
  Is it because the algorithm is actually outputting the symbols ("0", "3", etc...) which

are typically used to represent integers?
Is it because you could modify the algorithm so that, in each instance when it runs, the output is a sequence of natural language sentences which would be indistinguishable from those written by a mathematician presenting the definition of a certain finite sequence of integers?
I THINK THAT THERE IS SOMETHING PRIMITIVE GOING ON HERE.

- p.92 How to think of an embedding of an isomorphic copy of something inside of something lese (for example $\mathbb{Z} \subset \mathbb{R}$).

E.3 Enderton's *Elements of Set Theory* book

- "collection" is primitive.
- p.1 The intuitive natural numbers are taken as primitive.
- p.1-2 Human level true, false, not, if, every, exists.
- p.2 Equals, properties, statements.
- p.5-6 When a predicates is "meaningful" is left open-ended.
- p.10 Axioms, consequences
- p.11 Set, member are primitive.
- p.12 "logical consequence", "assignment of meaning".
- p.12 Reiterating that these are primitive: every, exists, not, if, true, false, etc...
- p.13 Defining symbols and notation.
- p.13 **CRUCIAL:** Defining notational usage of certain strings from formal logic as **SHORTHAND/NOTATION** for English sentences.
  CRUCIAL: BUT WE ARE NOT WORKING INSIDE A FORMAL LANGUAGE!!!
- p.13-14 Usage of parentheses as a primitive thing.
  Groupings for extracting meaning iand avoid ambiguity.
- p.22 **CRUCIAL:** Restricting the predicates that can be used in set-builder notation to those whose English sentences can be expressed in the formal language...
  CRUCIAL: FIXME::TODO What does this mean?
- p.36 Kuratowski def of ordered pairs.
- p.41 n-tuples as ordered pairs of ordered pairs.
- p.42 top. Again, the intuitive naturals are taken as primitive to define n-tuples.

- p.47 Composition of functions theorem $(F \circ G)(x) = F(G(x))$
- p.49-p.55 Enderton explicitly accepts the Axiom of Choice.
- p.68 Inducive sets, Infinity Axiom.
- p.73 Recursion theorem. Gives a correct proof. Mentions a bogus proof (like the one in Beck's *The Art of Proof* book).
- p.79-81 Common notation for things like $m + n$, $m \cdot (n + p)$, etc.
- p.79 **CRUCIAL: Interpretation of parentheses in Theorem 4I !!!**
  Symbol rewrite or substitution.
  Parentheses used to indicate and disambiguiate meaning.
  CRUCIAL: Don't forget that what you see on the page are symbols, and you as a human have to interpret them. The author uses parenthese to tell you how to do that.

E.4 Munkres' *Topology* book

- p.4 Symbols and objects.
- p.4 Sameness in the Leibnitx/Tarski sense.
- p.4 Equality as logical identity: The symbols refer to the "same" object.
- p.4-15 Basic logic (and, or, not, every, at least one, De Morgan's), naive sets (unions, intersections, subsets, etc).
  Ordered pairs as primitive.
- p.15 Functions as sets of ordered pairs.
- p.35 Hidden "inductive" definition of a^n.
- p.47 Not all English definitions can be used as predicates.
  Barber's Paradox (same as Russel's).
- p.46-47, p.52-55 Recursive/inductive definitions and calculations.
  Proof of Recursion Theorem.

E.5 Leon Henkin's article *On Mathematical Induction*

`https://www.jstor.org/stable/2308975`

This article has basically the same content as the section in Enderton on the Recursion Theorem.

Dedekind in *Was Sind Und Was Sollen Die Zahlen?*

> (Paraphrased) "The *axiom* of mathematical induction does not by itself justify *definitions* by mathematical induction."