Note To Self: Math Bootstrapping Notes v2

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Contents

0 Intro

TODO

0.1 What is this, who is it for, and why did I write it?

TODO

0.2 What problem am I trying to solve?

TODO
Where's the fun in informal mathematics (which includes pure math) if you can't imagine the things you're playing with as being real?

0.2.1 My Math Experiences

TODO

0.3 Some good quotes

If real is what you can feel, smell, taste and see, then 'real' is simply electrical signals interpreted by your brain.

-Morpheus, *The Matrix*

The axioms [concerning primitive notions] can be thought of as divulging partial information regarding the meaning of the primitive notions.

-Herbert Enderton, *Elements of Set Theory*

Young man, in mathematics you don't understand things. You just get used to them.

-John von Neumann

It depends upon what the meaning of the word 'is' is.

-President Bill Clinton

Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.
(God created the integers, everything else is the work of man.)

-Leopold Kronecker

Impeccable definitions have little value at the beginning of the study of a subject.

-Elliott Mendelson, *Introduction to Mathematical Logic*

We cannot expect that the cognizance of the natural number sequence can be reduced to that of anything essentially more primitive than itself.

-Stephen Cole Kleene, *Introduction to Meta-Mathematics*

The formally axiomatized propositions of mathematics cannot constitue the whole of mathematics; there must also be an intuitively understood mathematics. ... The metatheory belongs to intuitive and informal mathematics. ... The metatheory will be expressed in ordinary language.

-Stephen Cole Kleene, *Introduction to Meta-Mathematics*

Reductio ad absurdum [Proof by Contradiction], which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.

-G.H. Hardy, *A Mathematician's Apology*

> It seems obvious, for instance, that the formula:
>
> $$3 = 2 + 1$$
>
> is a true assertion, and yet some people are somewhat doubtful as to its truth. In their opinion, the formula appears to state that the symbols "3" and "$2 + 1$" are identical, which is obviously false since these symbols have entirely different shapes, and, therefore, it is not true that everything that may be said about one of these symbols may be said about the other.
> ...
> In order to avoid doubts of this kind, it is well to make clear to oneself a very general and important principle upon which the useful employability of any language is dependent. According to this principle, whenever, in a sentence, we wish to say something about a certain thing, we have to use, in this sentence, not the thing itself but its name or designation.
> ...
> The problem arises as to how we can set about to form names of words and expressions. There are various devices to this effect. The simplest one among them is ... quotation marks.
>
> -Alfred Tarski, *Introduction to Logic*

> (Paraphrased)
> The *axiom* of mathematical induction does not by itself justify *definitions* by mathematical induction.
> -Dedekind, *Was Sind Und Was Sollen Die Zahlen?*

0.4 Where is the boundary?

TODO: Figure out how to draw a nice boundary so that the approach cannot be faulted.

Make this document bulletproof.
But acknowledge that the primitive ideas are kind of circular (for example implicit universality when definintg notation maybe???).

BRYAN – Make this document lie within the intersection of the various mathematical philosophies.

You should be able to choose your own philosophy to work in, much like you can choose whether or not you want to work in a context allowing Axiom of Choice.

1 The Obvious Requirement: Basic Cognitive Skills

TODO

See https://en.wikipedia.org/wiki/Human_intelligence
See https://en.wikipedia.org/wiki/Cognition
See https://en.wikipedia.org/wiki/Cognitive_skill

Roughly at the level of a secondary (US - high school) education.
Maybe so obvious that nobody mentions it.
Required for pretty much everything in life.
Most disciplines involving analytical thinking are built on top of this.

The learning of math, CS, logic is separate from and sits on top of these.
But probably can be further refined by feedback and interplay with learning of math, CS, logic.

Fundamental human cognitive abilities, used in everyday life, not specific to math, but which should be present in order for the rest of the document to be understandable. These are innate human abilities which seem present to one degree or another in most people and which need to be developed to a certain degree before it makes sense to study pure math. Some appear to also be present in animals.

Some combination of nature and nurture, can be trained I guess.

Ability to read this document.
These skills might just be habits or automatic reactions.
Can be trained, refined, and improved.
There is a barrier to going deeper into how they work or what they really are?
How does the brain work? Are we alive? Are we actually thinking and making choices, or are we just somehow aware of some automatic brain activity? Playing violin vs focusing on how exactly you put your finger down.
Touch your index finger to your thuimb.
Are you actually controlling it, and if so, how are you doing it?

What level are we going to work at here?

For “Basic Cognitive Skills” Implication elimination and Process of elimination, have to be very careful to avoid traps!
Explicitly mention that we are going to avoid the loaded/charged words “true” and “false” until we start the math-specific stuff.
At the “Basic Cognitive Skills” level, let’s use words like context/option/possibility/situation/right/correct.
Don’t talk about “statement P being true”.
Don not work with abstract statements (but can work with concrete statements using abstract/invented entities).
Work only with explicit examples like “you know either the flower is red or the flower is blue, and you know it’s not red” and “you know if the flower is red, then the flower is a rose, and you also know that the flower is red”.

Want to decouple the the two forms of reasoning from what kind of objects/statements you're allowed to apply them to (too messy). Need to keep it as simple as possible.
Where's the cookie examples:

1. Process of Elimination: Two plates covered with foil, there's a cookie on least one of them.
2. Implication Elimination: Two plates covered with foil, you know that if the first plate is empty then there's a cookie on the second place.

Are these "equivalent"? Can't worry about it yet. That requires details of "meanings" of statements and their negation, possibly Excluded Middle or canceling double negatives.
These kinds of reasoning apply once you already **BELIEVE** that you have a guarantee/promise of some kind.
First you believe a promise/guarantee, then you base future beliefs on it.

Process of Elimination is:
1) Listing out possibilities
2) SOMEHOW knowing that all possbilities are included
3) Crossing out ones that can SOMEHOW be ruled out until only one remains.
4) Believing that the actual situation is single remaining possibility.

Implication Elimination is:
1) The belief that a promise based on a contingent condition ("if this occurs then that is guaranteed to occurs") is solid. (REPHRASE THIS).
2) SOMEHOW knowing that the contingent condition has occurred. (REPHRASE THIS).
3) Believing that the situation guaranteed by the promise has therefore also occurred. (REPHRASE THIS).

1.1 Level 0

The innate ability to acquire cognitive capacities, including but likely not limited to:

- attention, perception, memory, learning, pattern recognition, perceiving dichotomies, perceiving associations,
- counting and ordering,
- reasoning,
- language,
- categorizing, understanding hierarchies,
- imagination, metacognition, recognizing possibilities

1.2 Level 1

- Ability to perceive analogies and learn inductively from context
- Ability to communicate in some natural language
- Ability to perform basic counting and ordering for small finite collections of things
- Ability to understand dichotomies such as yes/no, before/after, same/different, blue/not-blue, smaller/larger, is/is-not
- Ability to reason by Implication elimination in simple concrete situations
  TODO: Give an example.
- Ability to reason by Process of elimination in simple concrete situations
  TODO: Give an example.
- Ability to perceive context ("implicit situational information", "ambient information")
- Ability to understand hierarchies (https://www.npr.org/sections/ed/2014/08/22/341898975, https://en.wikipedia.org/wiki/Hierarchy)
- Ability to follow simple instructions (perform a simple sequence of steps in order)

1.3 Level 2

- Ability to use and interpret symbols
- Ability to understand symbol substitution
- Ability to generalize and think abstractly.
  This includes the ability to determine which components of a situation are essential and which are purely cosmetic.
- Ability to imagine objects/contexts/possibilities
- Ability to recognize ambiguity
- Ability to recognize contradiction
- Ability to recognize self-reference
- Ability to engage in metacognition (including drawing a line between what you know and what you don't know)
- Ability to self-teach, verify work, and recognize mistakes
- Ability to use trees/parentheses to specify instructions and/or remove ambiguity
- Understanding that meaning can depend on context

- Possession of ”enough” real-life experience
- The ability to read this document
- Ability to reason by Implication elimination in simple but possibly abstract situations
  TODO: Give an example.
- Ability to reason by Process of elimination in simple but possibly abstract situations
  TODO: Give an example.
- The ability to engage in short and straightforward, but possibly abstract, deductive arguments by stringing together applications of Implication elimination and Process of elimination.
  TODO: Give an example.

2 Primitive concepts

TODO p.11 Axioms about primitive notions divulge ”partial meaning” of the primitive notions.

2.1 Things/objects/entities

TODO

2.1.1 What something “is” versus what we allow ourselves to do with it?

TODO
Python “is” vs “==”

2.1.2 What are mathematical objects, and do they really exists?

TODO
Mathematicians use parentheses (or trees) and symbol substitution all over the place to indicate how to do a calculation.
What does this say about how they think about mathematical objects?

2.2 Truth/Falsity/Negation

TODO

2.3 Context

TODO

2.4 Meaningful

TODO

2.5 Properties, statements, and predicates

TODO

2.6 Deductive reasoning

- Ability to engage in a multi-step deductive argument (a "proof") using implication elimination, process of elimination. Understanding of subproofs used in process of elimination.

- implication elimination, process of elimination, and variations of those (e.g. proof by contradiction and proof by cases).
  Understanding of nested proofs ("subproofs").
  Example: Proof $[(P \implies Q) \text{ AND } (Q \implies R)] \implies (P \implies R)$
  Example: Prove that Proof By Cases works.

- Imagining new contexts by adding information (adding then discharging assumptions).

- Step-by-step deductive reasoning

- Reasoning with implication elimination and process of elimination is like exploring a maze where some intersections might marked with reliable arrows saying "this way out", and also being able to add your own arrows when you discover dead ends.

- For thigns to be sane, you have to assume that contexts don't suddenly and mysteriously change underneath you while you're reasoning within one. Even if the target of your reasoning is some kind of evolving system...you then just have to make your context "bigger".

- (Beck's book) "unexamined basis of logic" NOTES: The human-level "unexamined basis of logic" natural way of thinking is good enough, but you have to be experienced enough to know what other peple would generally consider to be valid, and what the general "rules of the game" are.

- Proceed rigorously from intuitive primitive notions NOTES: We can be intuitive and informal about the foundations of math from (A) logic/set theory. But, to then proceed to build the structure of mathematics, we have to then rigorously apply the democratically accepted logical thought processes available as basic human capabilities. In math, there are always going to be undefined words which have only intuitive meanings (for example "collection" or "property").

- What happens if things *CHANGE* while I'm reasoning about them?

2.7 Universality, Applying universality, Proving universality

TODO

2.8 Existentiality, Applying existentiality, Proving existentiality

TODO

2.8.1 Short-circuiting universality and existentiality

TODO

2.8.2 Universality/existentiality duality

TODO

2.9 Impossibility

TODO

2.10 Guarantees/implies/promises

TODO

2.11 Equal/Identical

TODO

2.12 What things "are" versus how things "behave", characterizing properties

TODO p.11 Axioms about primitive notions divulge "partial meaning" of the primitive notions.

2.13 Collections, Informal rules for working with collections

TODO

2.14 Even more about universality and existentiality

TODO

2.14.1 Hidden implications in statements involving universality or existentiality

TODO

2.15 Ordered pairs as primitive?

TODO

2.16 The natural numbers

TODO

3 Some non-primitive concepts

TODO

3.1 unique, no more than one, at most one

TODO

3.2 Proof by Contradiction

TODO

3.3 Proof by Cases

TODO

3.4 implication/convese/inverse/contrapositive

TODO

3.5 Equivalence of an implication with its contrapositive

TODO

3.6 ordered pairs

TODO

3.7 relations

TODO

3.8 domain/range

TODO

3.9 functions

TODO

3.10 Some notation used in the context of relations and functions

TODO

3.11 Warning: Symbols are not functions!

TODO

3.12 Relations as graphs

TODO

3.13 Composition of relations

TODO

3.14 Composition of functions

TODO

3.15 Recursion Theorem

TODO

3.15.1 Inductive/recursive definitions

Dedekind in *Was Sind Und Was Sollen Die Zahlen?*

> (Paraphrased) "The *axiom* of mathematical induction does not by itself justify *definitions* by mathematical induction."

3.15.2 Algorithms (effective procedures)

TODO

3.15.2.1 Addition Algorithm (from Beck) TODO

3.15.2.2 Euclidean Algorithm (from Velleman) TODO

3.16 More about composition of relations and functions

TODO

3.16.1 Calculating a composite of functions in practice

TODO

3.16.1.1 All three ways of calculating a composite agree

TODO

4 Some math jargon

TODO

4.1 “holds”

TODO

4.2 if A is true then B is true

TODO

4.3 A is true if and only if B is true

TODO

4.4 A is true except if B is true

TODO

4.5 neither A nor B are true

TODO

4.6 vacuously true

TODO

4.7 X is of the form ...

TODO

4.8 Y depends only on X, Y constant with respect to Z

TODO

4.9 Y doesn't depend on Z, constant with respect to Z

TODO

4.10 Z can be solved for X in terms of Y, X is defined implicitly by Z and Y

4.11 unique up to (whatever)

4.12 Identify X with an isomorphic copy in Y

TODO
Beck p.92 How to think of an embedding of an isomorphic copy of something inside of something else (for example $\mathbb{Z} \subset \mathbb{R}$).

4.13 discharging an assumption

TODO: See if like popping a stack frame off the stack?
See `https://math.stackexchange.com/questions/3527285/what-does-discharging-an-assumption-`

Appendices

A About symbols, names, and notation

TODO

A.1 Naming things using symbols/handles

TODO

A.2 Names for things vs. the things themselves

TODO

A.3 Additional examples

TODO

A.4 Defining new names, words, symbols, shorthand, and notation

TODO

A.5 More about symbols

TODO

A.5.1 Implicit "for all" quantifier when defining notation???

TODO

A.5.2 "Composing" symbols???

TODO

A.5.3 Notational symbols as shorthand vs symbols in a formal language

TODO

B Supplemental Stuff

TODO

B.1 Relationship between *Implication Elimination* and *Process of Elimination*

TODO

B.2 De Morgan Duality: Existentiality vs. Universality

TODO

B.3 De Morgan Duality, Again: Unions vs. Intersections

TODO

B.4 Existentiality/Universality Distributive Behavior

TODO

B.5 Union/Intersection Distributive Behavior

TODO

B.6 Statements with More Than One Universal and/or Existential

TODO

B.7 Disambiguating the Word "And"

TODO

B.7.1 More about disambiguating the word “and”

TODO

B.8 Disambiguating the Word “Or”

TODO

B.9 Disambiguating the Words “Unless” and “Except”

TODO

C What did I look at?

- Enderton’s *Elements of Set Theory* book
- Velleman’s *How to Prove It* book
- Beck’s *The Art of Proof* book
- Munkres’ *Topology* book

C.1 Velleman’s *How to Prove It* book

- p.8 Basic reading ability, which requires short (ordered) lists of words and sentences.
- p.8 Introducing basic mathematical deductive reasoning.
- p.8 Statements, truth values, truth, falsity.
  CRUCIAL: These are intuitive, human-level statements, truth, falsity.
- p.8 Argument = sequence of steps in deductive reasoning.
  CRUCIAL: Requires intuitive understanding of small natural numbers to get short (ordered) list of steps in the argument.
- p.8 **CRUCIAL:** Section starting with ”But is this conclusion really correct”.
  Applying human-level reasoning to check validity of a deductive argument.
  Looking at the deductive argument and involved statements as if they were a things/objects/situations, and then reasoning about it ”from above”/”observer’s vantage point”.
- p.8 ”In this case, ...”: Imaginary scenario human reasoning.
- p.8 Process of elimination and implication elimination inside deductive reasoning, vs applying human-level elimination and implication elimination to check the validity.
- p.9 *Defining* the word *“valid”* as a *property* of an argument.

- p.9 Saying that something is *not valid* because it does not have the required property to be valid.

- p.9 Small natural numbers.

- p.9 Things like "P" and "Q" "stand for" "statements".
  CRUCIAL: This is completely informal.

- p.9 Informal notion of the *form* of an argument.

- p.11 Parentheses. Saying it's similar to parenetheses in algebra.

- p.12 strings = finite sequences.
  Translating human-readable argument into string of some form is *primitive.*

- p.12 uses the word "meaningless".

- p.15 Truth tables to capture possibilities.
  But reason about the truth tables by using human-level elimination and implication.

- p.15 Making a truth table requires being able to list out the members of $\{true, false\}^n$, and then fill in values for the calculated colums.

- p.17 Mentally parsing simple logical formulas.

- p.17 Using truth table to show argument is valid.

- p.19 Examining or reasoning with truth tables requires basic human-level counting and human-level reasoning (if-then, at least one, both, all, exists, etc...).
  Also (p.16-p.19), filling out a truth table or reasoning out the truth value of a composite statement is a human-level step-by-step process.
  CRUCIAL: This means human-level step-by-step processes are primitive.

- p.23 It appears to me from his usage of parentheses in formulas and the way he is constructing truth tables, that parsing out information from parenthesized human-level expressions is supposed to mentally happen in a step-by-step way using some primitive mental framework which has something like expression trees.

- p.26 Variables are symbols, and symbols "represent" things.

- p.28 collection, elements, explicitly specifying sets using braces, set-builder notation.

- p.29 Reasoning and drawing conclusions about sets appears to involve thinking about them the same way as if they were actually tangible, real-life objects.
  You are allowed to fix a mathematical entity in your mind and then reason about it in a step-by-step human way.

- p.31 Definition of truth set is using the subset specification axiom.

- p.32 Assume we have these sets of numbers, can reason about them as if they were real tangible objects and everyone agrees about what precisely they are.

- p.32 *context, universe of discourse, range over*

- p.35 He is assuming that we know how symbols/notation/shorthand work at a human level in order to define symbols for intersection and union of sets.
  Note that a lot of times, the meaning of a new symbol is defined in terms of other symbols, and the meaning of new notation is defined in terms of other notation.

- p.58 human-level every, at least one, and definition of quantifiers in terms of these.

- p.173 Ordered pairs is primitive for the author.

- p.234 **CRUCIAL:** Look very closely at this proof, especially the reasoning at the end for
  $(g \circ f)(a) = c = g(b) = g(f(a))$.
  "Equality", "let".
  Reasoning about the mathematical objects as if they are are unchanging real-life objects.

- p.293-294 Informal definition of recursion and recursively defined functions.
  No attempt to formally justify.
  Says basically "because we can compute $f(n)$ for any n, we really do have a function."
  Basically taking it as an axiom that this works and that you really get a function defined on the totality of the natural numbers. This is better than in Beck's book p.35, where there is a nonsense fake proof of the Recursion Theorem given, the exact type of proof that Enderton calls out in his set theory book on p. 76.

- p.294 Definitions with ellipses ("...") are often secretly recursive definitions.

- p.326-327 Euclidean Algorithm for finding gcd(m, n).

- p.328 The Euclidean Algorithm is used in a proof that gcd() is a linear combo of arguments.
  The Euclidean Algorithm produces a sequence of values.

- p.372 Using the natural numbers to define *finite*.

- p.383 Definition of finite sequence as a function on $\{1, 2, 3, \ldots, n\}$.

C.2 Beck's *The Art of Proof* book

- p.xvi "assumed to know" something" ?=? "assumed to have a working understanding of how more experienced mathematicians think of" something ???

- p.xvi Understanding of mathematical "truth" has evolved over a period of two thousand years?

- p.4 set is primitive.
- p.4 A ”distinguished” set of integers satisfying some axioms is taken as primitive.
- p.4 Parentheses determining ”order of operations” interpreted as ”do what’s on the inside first”.
  I think this means that step-by-step calculations are considered primitive.
- p.5 ”same number” ===¿¿¿ thinking of math entities in the way you would think of more concrete and real-life objects?
- p.5 Punting of ”truth”.
  ”True” if it can be deduced from other ”true” things. Interpreting the ”truth” of a statement in a particular context as one which can be used as a basis for further deductions.
- p.6 A ”proof” is a list/sequence of statements where each statement follows logically from previous ones.
- p.7 properties and predicates are primitive, and informal.
- p.10 Defining new binary operation (subtraction) and notation for that operation.
- p.11-12 There is an ”unexamined basis of logic” which works at the human level, and is required to even make the kinds of philosophical arguments about ”what is valid logic” (resulting in the ”examined logic”) that people have been wrestling about for 2000 years.
- p.12 The human-level ”unexamined basis of logic” natural way of thinking is good enough, but you have to be trained enough to know what other peple would generally consider to be valid, and what the general ”rules of the game” are.
- p.12 Not possible to state all of the axioms of thought or the universe that we are assuming in order to do math.
- p.12 Distinction between (A) logic/set theory, and (B) math based on logic and set theory.
  Necessary compromise in order to not get stuck on philosophy:
  We can be intuitive and informal about the foundations of math from (A) logic/set theory.
  But, to then proceed to build the structure of mathematics, we have to then rigorously apply the democratically accepted logical thought processes available as basic human capabilities.
- (NOTE FROM BRYAN) I think that there is a good analogy between musical training and mathematical training.
  What counts as ”generally acceptable, logical, basic human-level reasoning” is something learned by repetition, training, and pattern recognition.
  This feels similar to how you learn to play an instrument or where musical phrases are

going.
If this is the case, then maybe it's OK to think of it as an intuitive thing which you can enjoy and not stress out about too much.

- p.15 Assuming Excluded Middle as a basis from Proof By Contradiction.
- p.18 Assuming Induction Axiom for the naturals.
- p.18 Again, assuming that there is a (informal) primitive notion of "properties".
- p.25 These are informal and primitive: for all, there exists, and, or, if ... then (guarantees), impossible, not, etc.
- p.26 quantifiers
- p.31 In math, there are always going to be undefined words which have only intuitive meanings (for example "collection" or "property").
- p.35 Baby Recursion Theorem.
  CRUCIAL: THE PROOF IS BULLSHIT. ENDERTON SET THEORY CALLS OUT FAKE PROOFS OF THE RECURSION THEOREM ON p.76.
  CRUCIAL: Alternatively, you could look at it from the perspective that a mathematician might think that the proof is "good enough".
- p.52 "ordered pair" is primitive.
- p.63 "unique up to (whatever)" needs explanation.
- p.69 Algorithm is primitive and informal. A step-by-step procedure which is obviously unambiguous and mechanical.
- p.70 Elementary School Addition Algorithm:
  CRUCIAL: How to interpret this is super important!
  QUESTION: Why is it allowed to think of the outputs of the algorithm as integers?
  ANSWER: This is because there is an implicit application of the Recursion Theorem here.
  Beck should've shown what's really going on.
- p.92 How to think of an embedding of an isomorphic copy of something inside of something else (for example $\mathbb{Z} \subset \mathbb{R}$).

C.3 Enderton's *Elements of Set Theory* book

- "collection" is primitive.
- p.1 The intuitive natural numbers are taken as primitive.
- p.1-2 Human level true, false, not, if, every, exists.

- p.2 Equals, properties, statements.
- p.5-6 When a predicates is “meaningful” is left open-ended.
- p.10 Axioms, consequences
- p.11 Set, member are primitive.
- p.11 Axioms about primitive notions divulge ”partial meaning” of the primitive notions
- p.12 “logical consequence”, “assignment of meaning”.
- p.12 Reiterating that these are primitive: every, exists, not, if, true, false, etc...
- p.13 Defining symbols and notation.
- p.13 **CRUCIAL:** Defining notational usage of certain strings from formal logic as **SHORTHAND/NOTATION** for English sentences.
  CRUCIAL: BUT WE ARE NOT WORKING INSIDE A FORMAL LANGUAGE!!!
- p.13-14 Usage of parentheses as a primitive thing.
  Groupings for extracting meaning iand avoid ambiguity.
- p.22 **CRUCIAL:** Restricting the predicates that can be used in set-builder notation to those whose English sentences can be expressed in the formal language...
  CRUCIAL: FIXME::TODO What does this mean?
- p.36 Kuratowski def of ordered pairs.
- p.41 n-tuples as ordered pairs of ordered pairs.
- p.42 top. Again, the intuitive naturals are taken as primitive to define n-tuples.
- p.47 Composition of functions theorem $(F \circ G)(x) = F(G(x))$
- p.49-p.55 Enderton explicitly accepts the Axiom of Choice.
- p.68 Inductive sets, Infinity Axiom.
- p.73 Recursion theorem. Gives a correct proof. Mentions a bogus proof (like the one in Beck's *The Art of Proof* book).
- p.79-81 Common notation for things like $m + n$, $m \cdot (n + p)$, etc.
- p.79 **CRUCIAL: Interpretation of parentheses in Theorem 4I !!!**
  Symbol rewrite or substitution.
  Parentheses used to indicate and disambiguiate meaning.
  CRUCIAL: Don't forget that what you see on the page are symbols, and you as a human have to interpret them. The author uses parenthese to tell you how to do that.

C.4 Munkres' *Topology* book

- p.4 Symbols and objects.
- p.4 Sameness in the Leibnitz/Tarski sense.
- p.4 Equality as logical identity: The symbols refer to the "same" object.
- p.4-15 Basic logic (and, or, not, every, at least one, De Morgan's), naive sets (unions, intersections, subsets, etc).
  Ordered pairs as primitive.
- p.15 Functions as sets of ordered pairs.
- p.35 Hidden "inductive" definition of a^n.
- p.47 Not all English definitions can be used as predicates.
  Barber's Paradox (same as Russel's).
- p.46-47, p.52-55 Recursive/inductive definitions and calculations.
  Proof of Recursion Theorem.